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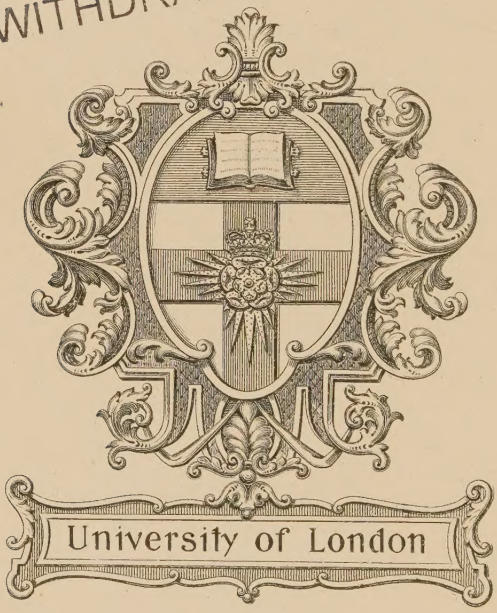

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. III.

FROM NOVEMBER, 1869, TO NOVEMBER, 1871.

LONDON:
C. F. HODGSON & SON, GOUGH SQUARE,
FLEET STREET.

ERRATA IN VOL. II.

p. 180, line 5, *for* $3r\,(p-2)^2$ *read* $3r\,(p-1)^2$.
p. 188, line 13 from bottom, *for* $-3m^2(r-1)\,(n-r)^2$ *read* $-3m\,(n-r)^2\,(r-1)\,(3r-1)$.

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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. III.

November 11th, 1869.

ANNUAL GENERAL MEETING, held at Burlington House, Prof. CAYLEY, F.R.S., President, in the Chair.

The Reports of the Treasurer and Secretaries were read and adopted.

From the Report of the Secretaries, it appeared that the number of members since the last General Meeting, held November 12th, 1868, had increased from 115 to 116.

The communications made to the Society during the past Session had been as follows:—

"On Certain Formulæ concerning the Theory of Discriminants; with applications to Discriminants of Discriminants, and to the Theory of Polar Curves:" by Dr. O. Henrici.

"On a Generalization of the Theory of Polars:" by Mr. W. K. Clifford.

"On Reducible Cyclodes &c.:" 3 papers by Prof. Sylvester.

"On Graphic Representations of the Disturbing Forces which act on Ships' Compasses, and the Deviations caused thereby" (Dygograms): by Mr. Archibald Smith.

"On a Correspondence of Points, such that a curve of the n^{th} order in one plane corresponds to a curve of the $4m^{th}$ order in another plane, with three multiple points of the order n in a straight line, and three other multiple points of the order $2n$:" by Mr. T. Cotterill.

"On the Mechanical Construction of Certain Quartic and Cubic Curves:" by Mr. S. Roberts.

"On Degenerate Forms of Conics:" by Prof. Hirst.

"On Tangents to the Cissoid:" by Mr. J. J. Walker.

"On the Analytical Formulæ which express the Deviations of the Compass:" by Mr. Archibald Smith.

"On the Rational Transformation between Two Spaces:" by the President.

"Properties of Homographic Figures:" by Prof. H. J. S. Smith.

"On an Envelope in the Cubic Correspondence of Points:" by Mr. T. Cotterill.

"On Series of Curves, especially on the singularities of their envelopes, with applications to Polar Curves:" by Dr. O. Henrici.

"Results arrived at by Quadric Transformation: by Prof. Hirst.

Lesser communications had been made by the President, Mr. Clerk-Maxwell, Prof. Smith, Mr. Clifford, Mr. Roberts, and Mr. Walker.

The same Mathematical Journals had been subscribed for as in the preceding Session, with the addition of the "Mathematische Annalen," edited by Messrs. Clebsch and Neumann.

The Proceedings of the Society had been sent to the Royal Societies of London and of Edinburgh; the Royal Irish Academy; the Cambridge Philosophical Society; the Philosophical Society of Manchester; the Institute of Actuaries; the British Museum; l'Académie des Sciences, Paris; l'Ecole Polytechnique; la Société Philomathique; Akademie der Wissenschaften, Berlin; il Istituto Lombardo, Milan; the Smithsonian Institution, Washington, U.S.; Prof. Luigi Cremona, an editor of the Annali di Matematica; Herr Borchardt, Berlin, editor of Crelle's Journal.

In return for these, the Society had received the Proceedings of the Royal Society; the Journal of the Institute of Actuaries; Memorie and Rendiconti of the Istituto Lombardo; the Annali di Matematica; the Monatsberichte for 1869, from the Akademie der Wissenschaften; Crelle's Journal, from Herr Borchardt; Proceedings and Memoirs from the Literary and Philosophical Society, Manchester.

The following presents had been made to the Society:—

"On graphic representations of the disturbing forces which act on ships' compasses, and the deviations caused thereby;" a copy of which was sent to every member at the author's (Mr. A. Smith's) expense.

"Société Philomathique de Paris. Extrait de la Séance du 18 Janvier, 1868:" from Prof. Hirst.

"Proceedings of the Literary and Philosophical Society, October, 1868," and "Recherches sur les surfaces réglées tétraédrales symmetriques, par Jules de la Gournerie:" from Prof. Cayley.

"On the Curves which satisfy given conditions," by Prof. Cayley: from the Author.

"A Photographic Diagram, consisting of 10,206 lines connecting 144

points on a circle, and involving 17,178,876 intersections, to illustrate Mr. Woolhouse's Question in the Lady's and Gentleman's Diary for 1867:" from Mr. Woolhouse.

"Mémoire de géométrie pure sur les surfaces du troisième ordre."

"Royal Society Catalogue of Scientific Papers," vol. ii. COA—GRA.

"Sulle Superficie Gobbe di quarto grado," by Prof. Cremona: from the Author.

"Étude sur le déplacement d'une figure de forme invariable," par A. Mannheim: from the Author.

"Mechanical Geometry," by A. H. L. S. Béchaux, B.A.: from the Author.

"Criticism on Mr. Wilson's paper "On Euclid as a text-book of Elementary Geometry," by Profs. Brioschi and Cremona: from the Authors.

"Ueber systeme von functionen mehrer variabeln," by Prof. Kronecker: from the Author.

"On Bicircular Quartics:" by J. Casey, B.A., from the Author.

"Intorno al numero dei moduli delle equazioni," by Profs. Cremona and Casorati: from the Authors.

"Sulla trasformazione delle curve iperellitiche," by Prof. Cremona: from the Author.

"Le relazioni fondamentali tra i moduli di periodicità degli integrali abeliani di prima specie," by Prof. Casorati: from the Author.

Cartes de visite of the following gentlemen had been presented for insertion in the Society's Photographic Album:—

Revs. R. Harley, H. Martin, Dr. Salmon, R. Townsend; Profs. Hirst and H. J. S. Smith; Dr. Ingleby; and Messrs. W. K. Clifford, A. De Morgan, G. C. De Morgan, A. J. Ellis, J. Clerk Maxwell, H. Perigal, W. Spottiswoode, I. Todhunter, R. Tucker, J. M. Wilson, and W. S. B. Woolhouse.

After the reading of the Reports, and the appointment of Mr. C. W. Merrifield, F.R.S., as Auditor for the ensuing Session, the meeting proceeded to the election of the new Council.

The following gentlemen were declared by the Scrutators to be duly elected:—

President, Prof. Cayley; Vice-Presidents, Messrs. A. De Morgan, W. Spottiswoode, and Prof. Sylvester; Treasurer, Prof. Hirst; Secretaries, Messrs. M. Jenkins and R. Tucker; other Members of Council, Messrs. W. K. Clifford, T. Cotterill, M. W. Crofton, O. M. F. E. Henrici, S. Roberts, J. Stirling, Archibald Smith, Prof. H. G. S. Smith, and J. J. Walker.

The Rev. James White, M.A., was proposed for election.

Votes of thanks were carried, to the Chemical Society for the renewed use of their rooms; to Dr. Sharpey for the continued use of shelves in his room at University College, (with reference to this vote, Mr. S. Roberts expressed the hope that at no distant date the Society would

have a room of its own for the purpose); and to the retiring members of the Council for their services.

Mr. Tucker then read the following communication from Mr. G. O. Hanlon, on

The Vena Contracta. By G. O. Hanlon, Esq.

The Vena Contracta has a very important bearing on the calculation of the momentum of water flowing through an orifice, and the omission of its consideration has given rise to mistakes in calculation which it is the object of this paper to point out. It was stated in a controversy on the "Nozzle Ship," about two years ago, in the "Engineer," by a gentleman well known for his scientific attainments, that when water flowed through an orifice in a tank, the pressure on the side of the tank opposite to the orifice became doubled. Nothing could be more erroneous, and I pointed out, in a letter to that journal, the error of such a supposition. The controversy, however, at the time had ended, so that my views on the matter, although published, were not commented on. I will begin by investigating the subject analytically, and will subsequently see how far my result agrees with observation.

I will proceed in my calculation as if the pressure remained the same as before the orifice was opened; which is clearly the case, since the only pressure on the opposite side, after the orifice has been opened, is the height of water above any area; and it is easy to see that the *motion* of the water through the orifice could not add to the *pressure* of the water, since it is that very pressure which produces the motion. Indeed, it may be taken as an axiom, that where a force produces any motion, it is impossible for that motion to add to the original force. The very idea contains the solution of the problem of perpetual motion; yet, simple as the axiom appears, it is frequently overlooked by writers on Mechanics; indeed, the error pointed out in the present paper is an instance of such oversight.

Let $x = fy$ be the equation to the curve formed by the surface of the rushing water in the plane of its least sectional area parallel to the plane of the orifice, where the axis of X is vertical, and that of Y is horizontal, the axis of X being wholly outside the orifice. Let a and b be the two values of x when y has equal roots, and let y' and y'' be the two values of y corresponding to any value of x. Then, the mass of rushing water varying as the velocity, the momentum will be represented by the square of the velocity multiplied into the area. The whole momentum will therefore be

$$\int_a^b (y' - y'')\, dx \,.\, 2gx.$$

This must equal the pressure in the opposite direction, which equals

gAC, where A is the area of the orifice, and C the distance from the surface to the centre of gravity of the orifice. But

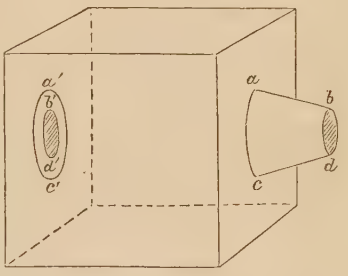
$$\int_a^b (y'-y'')\,x\,dx$$

equals the area of the curve $x=fy$ multiplied into the distance of its centre of gravity from the surface, which we will put equal to A′C′;

therefore $$g\mathrm{AC} = 2g\mathrm{A}'\mathrm{C}'.$$

If we suppose C=C′, we have A = 2A′.

Whether C always equals C′, or is always greater or less, is a result which probably can never be determined; but from actual measurement it appears that A=2A′, and if this is true it follows that C=C′. But it is certain that the greater the ratio between the height of water and the vertical height of the orifice, the more nearly do the centres of gravity coincide (*i.e.*, C=C′); and on this assumption we may conclude that, generally, the least sectional area of the flowing stream is half that of the orifice. This gives the coefficient of contraction equal to ·5, which agrees with the result obtained by actual measurement, and yet *was obtained on the supposition that no alteration of pressure took place when the water flowed through the orifice.* Now I will endeavour to show the reason of the above practically.



Let us suppose an orifice *ac* cut in the tank in the figure, and that the shape assumed by the flowing water is *ab*, *cd*, where the area of the section *bd* equals half that of the section *ac*. Now let us suppose a nozzle *abcd* to be fixed to the orifice, exactly the shape the water takes in flowing, and let us suppose the end of the nozzle *bd* closed, and the water at rest. Let *a′c′* and *b′d′* be the projections of the sections *ac* and *bd*. There is a horizontal pressure on *ab* and *cd* all round the side of the nozzle equal to the pressure on the section *bd*, since it is equal to the pressure all round *a′b′*, *c′d′*, which is by hypothesis equal in area to *b′d′*. Now let us remove the stopper *bd* at the end of the nozzle. The result is that the pressure ceases all round the nozzle, while the flowing water through *bd* gains that pressure which was round it, and consequently the pressure there is doubled; but that double pressure, since there is now no pressure from *ab*, *cd*, to give a corresponding pressure to *a′b′*, *c′d′*, has to be spread over *a′c′*, which is twice the area of *bd* or *b′d′*, and consequently this pressure on *a′c′* remains the same as before the stopper was taken from the nozzle.

Remarks on the preceding Paper, by Mr. J. Clerk-Maxwell, F.R.S.

It appears from this paper that it has been stated, that when a hole is made in one side of a vessel containing water, so as to allow the water to escape, the pressure on the opposite side of the vessel is thereby increased. This statement Mr. Hanlon shows to be erroneous, and asserts that the immediate effect of opening the hole, and allowing motion to take place, is to diminish the pressure in every part of the vessel.

He then shows, by a calculation of the momentum communicated in a unit of time to the fluid which is projected from an orifice in the side of a vessel, that the force acting on it is measured by $\iint \rho v^2 d\mathrm{S}$, where ρ is the density, v the velocity at the Vena Contracta where the pressure vanishes, and $d\mathrm{S}$ an element of the section of the stream.

Now at the Vena Contracta $v^2 = 2gz$ approximately, where z is the depth, so that the force of propulsion is $\mathrm{F} = 2g\rho \iint z d\mathrm{S}$; or if $\bar{z}$ is the depth of the centre of gravity of the section S, $\mathrm{F} = 2g\rho \bar{z}\mathrm{S}$. But at a distance from the hole, where the velocity is small, the pressure is the hydrostatic pressure $p = g\rho z$.

Let us assume (for an instant) that this is the value of the pressure on every part of the surface of the vessel, in which there is a hole whose section is S′, and whose centre of gravity is at a depth $\bar{z}'$. The resultant of the pressures on the sides will be $\mathrm{F}' = g\rho \bar{z}'\mathrm{S}'$, and this must be equal to F, because action and reaction are equal and opposite. This gives

$$2\bar{z}\mathrm{S} = \bar{z}'\mathrm{S}'.$$

There is very little difference between $\bar{z}$ and $\bar{z}'$, the depths of the centre of gravity at the hole and at the Vena Contracta. Hence we ought to have $\mathrm{S} = \frac{1}{2}\mathrm{S}'$, or section of Vena Contracta = ·5 of the hole.

I am not aware how far the author is original in this method of treating the subject; but it is certainly very clear and intelligible, without the aid of symbols, which is very important in practical matters. Rankine, however (Steam Engine, p. 96), gives ·6zS′ as the section of the Vena Contracta for round holes, and I have not seen any experimental result as low as ·5S′.

This arises partly from our having made an erroneous assumption,—that the pressure on the sides is the hydrostatic pressure ρgz, whereas it must be diminished by $\frac{1}{2}\rho v^2$ wherever there is a velocity v.

Now v will evidently be greatest near the hole, so that the pressure on the side next the hole will be diminished for this reason, as well as by the removal of the stopper of the hole, and the value of F′ will be greater than $g\rho \bar{z}'\mathrm{S}'$, and therefore S will be greater than ·5S′.

If the fluid is a perfect liquid originally at rest, it will have a velocity-potential ϕ, and the pressure is given by the equation

$$p = \rho gz - \rho \frac{\delta\phi}{\delta t} = \rho gz - \rho \frac{d\phi}{dt} - \tfrac{1}{2}\rho V^2,$$

where $\frac{\delta}{\delta t}$ denotes the rate of change in a moving particle, and $\frac{d}{dt}$ denotes rate of change at a point of space. V is the velocity of the fluid.

We must also have at the surface of the vessel $\frac{d\phi}{dv} = 0$, where v is the normal to the surface, and ϕ satisfies Laplace's equation

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0.$$

Now consider the case of a body which conducts electricity,—a mass of copper of the same form as the vessel; and let electricity be supplied at the upper surface, and carried off at the point corresponding to the hole, so that the current of electricity entering or escaping is everywhere proportional and parallel to the current of water. Then the electric-potential everywhere will be equal to ϕ, and its value is perfectly definite for a given problem, and problems can be constructed for which the solution can be found by our present methods.

Let ψ be the electric-potential for the copper conductor; then, if the form of the vessel and the motions of the surface remain similar, the velocity-potential $\phi = \psi T$, where T is a function of the time.

If we assume $\psi = 0$ at the free surface, then, since ψ increases along every line of motion, the value of ψ will be positive throughout the vessel, and will have its greatest values at the place of the issuing stream.

Let us first consider the *immediate* effect of opening the hole. In this case we may neglect the square of the velocity; and since at the orifice the pressure becomes zero, we have

$$p = 0 = \rho g\bar{z} - \rho \frac{d\bar{\phi}}{dt},$$

where $\bar{\phi}$ is the value of ϕ at the orifice. If $\bar{\psi}$ is the value of ψ at the orifice, then $\bar{\phi} = \bar{\psi} g\bar{z} t$;

and if ψ is the value of ψ at any other point, then

$$\phi = \psi g\bar{z} t,$$

and

$$p = \rho gz - \rho g\bar{z} \frac{\psi}{\bar{\psi}}.$$

Hence the assertion that the pressure is everywhere diminished is correct, and the amount of diminution is everywhere proportional to ψ, the electric-potential of a conducting mass similar to the water in the vessel, of which the upper surface is maintained at potential zero and the orifice at potential unity, the sides being supposed coated with insulating material.

When the motion has become constant, we may neglect $\frac{d\phi}{dt}$, and we have $$p = \rho g z - \tfrac{1}{2}\rho V^2,$$ where V is the velocity at any point. At the orifice, $V^2 = 2gz$. It appears, therefore, that in this case also the pressure at every point is less than the hydrostatic pressure.

Mr. Jenkins read a letter from Mr. Clerk-Maxwell, asking the following question :—

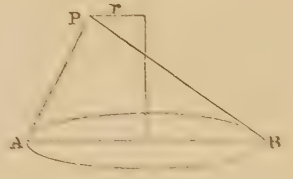
"Can the potential of a uniform circular disc at any point be expressed by means of elliptic integrals? Suppose V is the potential of the disc bounded by the circle

$$z = 0, \quad x^2 + y^2 = a^2;$$

then $$\frac{dV}{dx} = 2x\left(\frac{a}{r}\right)^{\frac{1}{2}} \frac{1}{\sqrt{c}}\,(E - F),$$

where $$r^2 = x^2 + y^2;$$

and if AB be a diameter parallel to r,



$$c = \frac{PB - PA}{PB + PA},$$

and E, F are complete elliptic functions for modulus c;

also $$\frac{dV}{dy} = 2y\left(\frac{a}{r}\right)^{\frac{1}{2}} \frac{1}{\sqrt{c}}\,(E - F).$$

"But $\frac{dV}{dz} = w$, where w is the solid angle subtended at P by the circle; that is, the area of the spherical ellipse on a sphere of unit radius cut off by the cone whose vertex is P and base the circle. We have expressions for $\frac{dV}{dx}$ and $\frac{dV}{dy}$. Can $\frac{dV}{dz}$ also be expressed by elliptic functions? and if so, can V itself be so expressed?

"I am writing out the theory of circular electric currents, in which these quantities occur. The expression $\frac{dV}{dz}$ for an elliptic disc can be found if we know it for a circular one, for the spherical ellipses in the one case are no more complicated than in the other. Can $\frac{dV}{dz}$, or V itself, be found for the elliptic disc?"

The President and Mr. Merrifield spoke on the question without deciding it.

Mr T. Cotterill read a paper on "Opposite Points on a Curve." Mr. Roberts and Mr. Clifford joined in a discussion on the paper.

The following presents were made to the Society :—

"Annali di Matematica," serie ii^a^, tomo iii^o^, Fas. 1^o^ Ottobre, 1869.

"Monatsbericht," Juli, August, 1869.
"Nature," Nos. 1 and 2.
"Royal Society Catalogue of Scientific Papers," vol. iii.

November 25th, 1869.

Prof. HIRST, and subsequently Prof. SYLVESTER, V.P., in the Chair.

The Auditor (Mr. Merrifield) certified that he had examined the Treasurer's accounts, and found them perfectly correct. The Treasurer then suggested that a statement of the financial position of the Society should be drawn up and circulated amongst the Members. This proposal was unanimously agreed to by the Meeting.

The Rev. James White was elected a Member, and admitted into the Society. The Rev. Percival Frost, M.A., was proposed for election.

Dr. Henrici exhibited a model of the cubic surface

$$xyz-(\tfrac{3}{7})^3(x+y+z-1)^3=0,$$

which has three biplanar nodes (to a scale of $2\frac{1}{2}$ inches as unit). A sufficient number (eleven) of sections, $x+y+z-1=$ a constant, cut out in cardboard, are connected in a horizontal position, and kept at their proper distance by three vertical sections $y=z$, $z=x$, $x=y$, with regard to which the surface is symmetrical. The model contains the central part of the surface with the three nodes, and is bounded by a sphere of eight inches' radius, with its centre at the origin, large enough to show the position of the three straight lines in the surface (each counting for nine), and to give an idea how the surface extends to infinity. The interstices between the cardboard are intended to be filled up with plaster of Paris, so as to form a solid model.

Mr. W. K. Clifford then read the following paper:—

On Syzygetic Relations among the Powers of Linear Quantics.

In his *Géométrie de Direction* (Paris, 1869), M. Paul Serret makes very beautiful use of a principle which he states nearly as follows (p. 138):—

"*In order that a system of points (in a plane) may be so related that every curve of order m passing through all but one of them must pass through the remaining one, it is necessary and sufficient that the* m^{th} *powers of their distances from an arbitrary line should satisfy a linear homogeneous relation* $\lambda_1 P_1^m+\lambda P_2^m+\lambda_3 P_3+\ldots\ldots\equiv 0$." *

There is, of course, an analogous theorem for surfaces, and in fact M. Serret combines the two enunciations into one; he states also the correlative theorems concerning a system of lines or planes such that

* In the *Bulletin des Sciences Mathématiques et Astronomiques*, January, 1870, M. Darboux observes that this theorem, for the special case $m=2$, had been given by Hesse, *Vier Vorlesungen aus der analytischen Geometrie*, Leipzig, 1866.

every curve or surface touching all but one of them, touches also the remaining one. For the sake of clearness I have here stated in full only one of these four theorems.

By the use of Professor Sylvester's method of Contravariant Differentiation I have arrived at certain extensions of these theorems, which I now proceed to explain:—

Theorem I. *In order that a system of* N *points in a plane should all lie on a curve of order* n, *it is sufficient that the* p^{th} *powers of their distances from an arbitrary line should satisfy a linear homogeneous relation; the number* N *being given by the formula*

$$N = \frac{\alpha}{2} n(n+3) + \tfrac{1}{2}(\beta+1)(\beta+2),$$

where α *is the quotient and* β *the remainder of the division of* p *by* n, *so that* $p = \alpha n + \beta$, *and* $\beta < n$.

Theorem II. *In order that a system of* N *points in space should all lie on a surface of order* n, *it is sufficient that the* p^{th} *powers of their distances from an arbitrary plane should satisfy a linear homogeneous relation; the number* N *being given by the formula*

$$N = \frac{\alpha}{6} . n(n^2+6n+11) + \tfrac{1}{6}(\beta+1)(\beta+2)(\beta+3),$$

where as before $p = \alpha n + \beta$, $\beta < n$.

To render the nature of these theorems somewhat more clear, I add the following tables of the values of N for given values of p and n:—

Table A.—Curves.

Values of p.	2	3	4	5	6	7	8	9	10	11	12
Line	5	7	9	11	13	15	17	19	21	23	25
Conic	6	8	11	13	16	18	21	23	26	28	31
Cubic		10	12	15	19	21	24	28	30	33	37
Quartic			15	17	20	24	29	31	34	38	43
Quintic				21	23	26	30	35	41	43	46
Sextic					28	30	33	37	42	48	55
Septic						36	38	41	45	50	56
Octavic							45	47	50	54	59

Table B.—Surfaces.

Values of p.	2	3	4	5	6	7	8	9	10	11	12
Plane	7	10	13	16	19	22	25	28	31	34	37
Quadric	10	13	19	22	28	31	37	40	46	49	55
Cubic		20	23	29	39	42	48	58	61	67	77
Quartic			35	38	44	54	69	72	78	88	103
Quintic				56	59	65	75	90	111	114	120
Sextic					84	87	93	103	118	139	167
Septic						120	123	129	139	154	175
Octavic							165	168	174	184	199

Here, for example, in the first table opposite the word Cubic and under the power 5 we find the number 15; the theorem corresponding to this is—

If 15 points are such that every quintic through 14 of them passes through the remaining one, all these points must lie on a cubic curve.

Now if we take 15 points arbitrarily on a cubic curve, it is not in general true that the fifth powers of their distances from an arbitrary line satisfy a linear homogeneous relation. That this may be the case, the 15 points must be intersections of the cubic with a quintic; and these are not arbitrary points, but 14 of them being given, the 15th is determined, by a theorem of Jacobi and Plücker. The theorem immediately derived from the table, then, must be completed by this statement; the points are not only all on a cubic, but they are intersections of a cubic and a quintic.

It is to be understood also that if we take a number N of points lying between any two adjacent numbers in the same vertical column of the table, then the same theorem is true about N that is true about the greater of these numbers. Thus we are informed by the first table that a syzygy among the 4th powers of the distances of 12 points makes them lie on a cubic, and that a similar syzygy for 15 points makes them lie on a quartic; this latter theorem is true for the intermediate numbers 13 and 14. It is not however *all* that is true in either of these cases; the 14 points are points of intersection of two quartics, and the 13 points are (I believe) points on a cubic such that no twelve of them are intersections of the cubic with a quartic. I wish particularly to draw attention to these intermediate cases, where it appears that more is true than can be proved by the method to be presently explained.

Method of Demonstration. Let the tangential equation of a point be

$$0 = a\xi + \beta\eta + \gamma\zeta \ (\equiv p, \text{ say})$$

and let the equation of a curve of the n^{th} order be

$$0 = (*\,\mathcal{X}x, y, z)^n \ (\equiv \mathrm{B}_n, \text{ say})$$

then I say that

$$\left(*\,\mathcal{X}\frac{\delta}{\delta\xi}, \frac{\delta}{\delta\eta}, \frac{\delta}{\delta\zeta}\right)^n . (a\xi+\beta\eta+\gamma\zeta)^n = (*\,\mathcal{X}a, \beta, \gamma)^n \lfloor n ;$$

that is to say, *if we operate with* B_n *on the* n^{th} *power of* p, *we shall obtain the result of substituting the coordinates of* p *for* x, y, z *in* B_n. If, then, this result vanishes, the point p is on the curve B_n.

I will now prove that if the 12th powers of the *nil-facta* in the tangential equations of 43 points are connected by a linear syzygy, the 43 points are on a quartic curve. We can draw a quartic B_4 through 14 of the points; operate with B_4 on the syzygy, then these 14 points are cleared away, and there remains a syzygy between the 8th powers of

the remaining 29 points. We have therefore now to prove that these 29 points are on a quartic. Draw a curve C_4 through 14 of them, and operate on the new syzygy with C_4. This clears away 14 more points, and we are left with a syzygy among the 4th powers of 15 points. But then by Serret's theorem these lie on a quartic. Hence, *any* 15 of the original 43 points are on the same quartic; therefore all the 43 are on the same quartic.

To prove that if the cubes of 13 points in space are connected by a syzygy, they lie on a quadric surface, operate with the plane through three of them; we are then left with a syzygy among the squares of 10 points, and Serret's theorem again applies.

The application of this method to the remaining cases will now be easy.

Mr. S. Roberts made some remarks on Invariants.

December 9th, 1869.

Prof. CAYLEY, President, in the Chair.

The Rev. Percival Frost, M.A., was elected a Member.

A Paper by Prof. H. J. S. Smith, "On the Focal Properties of Correlative Figures," was an Appendix to a former paper by the same author, "On the Focal Properties of Homographic Figures." By the term "Focal Properties" are intended those properties which arise from considering the imaginary circular points at an infinite distance in either figure, and the points corresponding to them in the other figure. These properties appear to be much less varied in their character in the case of two correlative figures than in the case of two homographic figures: and the two following theorems (of which the first is well known) will suffice to give an idea of the general nature of the results.

I. In two correlative figures in space, there are always two corresponding tetrahedra, such that three adjacent edges of each are rectangular; the three edges opposite to these being at an infinite distance, and the edges at a finite distance in either figure corresponding to the edges at an infinite distance in the other.

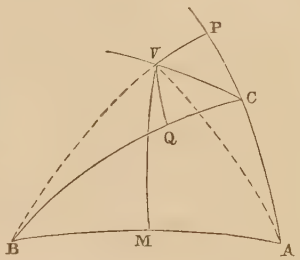
II. If we consider any point in either figure, and its correlative plane in the other, we have two definite planes passing through the point, and two corresponding points upon the plane, which may be called respectively the cyclic planes of the point, and the foci of the plane. If we take any third point in the plane, the angles which its focal radii vectores make with the line joining the foci are equal to the angles which the traces of the corresponding planes upon the cyclic planes make with the line of intersection of those two planes.

These theorems suppose only that in the two correlative figures the plane at an infinite distance in either figure answers to a point at a finite distance in the other.

Mr. Tucker read a proof, due to Mr. M. W. Crofton, F.R.S., of Gauss's Theorems and Napier's Analogies.

Bisect the side c at right angles by the arc MV, which meets the exterior bisector of the vertical angle C in V.

Draw the perpendiculars VP, VQ on the sides b, a of the triangle. Since VP=VQ, and VA=VB, and the angles at P and Q are right angles, the triangles AVP, BVQ are equal in all respects. Hence it will be seen that



$$\angle \mathrm{VAP} = \angle \mathrm{VBQ} = \frac{\mathrm{A}-\mathrm{B}}{2},$$

$$\angle \mathrm{VAM} = \angle \mathrm{VBM} = \frac{\mathrm{A}+\mathrm{B}}{2};$$

also, since CP=CQ,

$$\mathrm{BQ} = \mathrm{AP} = \frac{a+b}{2}, \quad \mathrm{CQ} = \mathrm{CP} = \frac{a-b}{2}.$$

Now the $\angle \mathrm{AVP} = \angle \mathrm{BVQ}$; adding AVQ to both, $\mathrm{QVP} = \mathrm{BVA}$, these two angles being bisected by VC and VM. Hence, in the right-angled triangles VMA, VPC,

$$\cos \mathrm{AM} \sin \mathrm{VAM} = \cos \mathrm{MVA} = \cos \mathrm{CVP} = \cos \mathrm{CP} \sin \mathrm{VCP},$$

that is, $\cos \frac{1}{2}c \sin \frac{1}{2}(\mathrm{A}+\mathrm{B}) = \cos \frac{1}{2}\mathrm{C} \cos \frac{1}{2}(a-b)$,

one of Gauss's theorems.

Again, considering the triangles AVP, CVP,

$$\sin \frac{a+b}{2} = \tan \mathrm{VP} \cot \frac{\mathrm{A}-\mathrm{B}}{2},$$

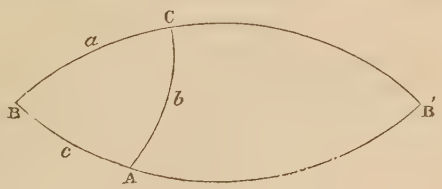
$$\sin \frac{a-b}{2} = \tan \mathrm{VP} \cot \tfrac{1}{2}(\pi-\mathrm{C});$$

dividing,

$$\frac{\sin \frac{a-b}{2}}{\sin \frac{a+b}{2}} = \tan \frac{\mathrm{C}}{2} \tan \frac{\mathrm{A}-\mathrm{B}}{2},$$

giving one of Napier's analogies.

The remaining formulas can be derived from the same construction, or an analogous one obtained by bisecting the angle C *internally*. But if one formula be proved, all the others follow. Thus, by applying the above formula of Gauss to the triangle AB′C, formed by producing a and c to meet, we have at once



$$\sin\frac{c}{2}\cos\frac{A-B}{2}=\sin\frac{C}{2}\sin\frac{a+b}{2};$$

and the remaining two follow from the polar triangle.

Mr. Crofton remarks, that "Prof. Gudermann, of Cleves, in his work on Spherics, gives a construction like mine, but deduces the formulas in a much more complicated form."

Mr. S. Roberts gave an account of his paper

On the Order of the Discriminant of a Ternary Form.

I propose to give a short analytical proof of two results obtained geometrically by Professor Cremona, relative to the influence of multiple points common to the curves of an involution. A few remarks are added on some limiting cases. The formula referred to will be found at pp. 261, 262 of the German translation of Professor Cremona's Introduction to the Geometrical Theory of Plane Curves (1865).

1. Let U, V be ternary forms of the n^{th} degree, so that $U+kV=0$ represents an involution of curves. In the general case, we obtain the number of possible double points of the involution, or the order relative to k of the discriminant, by means of the system of determinants

$$\begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \end{vmatrix} = 0.$$

Taking the two equations

$$\frac{du}{dx}\frac{dv}{dz}-\frac{du}{dz}\frac{dv}{dx}=0 \ldots\ldots (1), \quad \frac{du}{dy}\frac{dv}{dz}-\frac{du}{dz}\frac{dv}{dz}=0 \ldots\ldots (2),$$

we get the gross order $4(n-1)^2$, from which must be deducted the extraneous order of

$$\frac{du}{dz}=0 \ldots\ldots (3), \quad \frac{dv}{dz}=0 \ldots\ldots (4),$$

or $(n-1)^2$, leaving $3(n-1)^2$.

2. Suppose, however, that all the curves of the involution have a common multiple point at $x=0$, $y=0$ (say $\overline{xy}$) of the order of multiplicity p; and suppose, further, that the tangents at the point are common, and that s of them coincide with $y=0$. We may write

$$U = y^s a z^{n-p} + b z^{n-p-1} + \&c.,$$
$$V = y^s \alpha z^{n-p} + \beta z^{n-p-1} + \&c.$$

The highest power of z in (1), (2) is $2n-2p-2$, and its coefficient contains in each case y^{s-1}. These equations, therefore, represent curves having $\overline{xy}$ for a multiple point of the order $2p$, and there are $s-1$ common tangents; also (3), (4) represent curves having in common

a, b, c, &c., then the number of nodes is $= \frac{1}{2}(\Sigma a \,.\, \Sigma ab - \Sigma abc)$, or, as this may also be written, $\frac{1}{2}(\Sigma a^2 b + 2\Sigma abc)$. In particular, the formula applies to the case of the surface

$$\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix} = 0,$$

(a, b, c, d) being here the orders of A, B, C, D respectively, and the orders of F, G, &c., being $\frac{1}{2}(b+c)$, $\frac{1}{2}(a+c)$, &c. If the terms are all of them linear functions of the coordinates, or $a=b=c=d=1$, then the number of nodes is $=10$.

7. That the surface has nodes is, in fact, clear from the consideration that any point for which the minors of the determinant all vanish will be a node; and that (for the symmetrical determinant), by making the minors all of them vanish, we establish only a three-fold relation between the coordinates. The expression for the number of the nodes is, I think, obtained most readily as follows:

The nodes will be points of intersection of the curve and surface

$$\left\| \begin{matrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \end{matrix} \right\| = 0, \quad \begin{vmatrix} B, & F, & M \\ F, & C, & N \\ M, & N, & D \end{vmatrix} = 0,$$

these, however, contain in common the points

$$\left\| \begin{matrix} H, & B, & F, & M \\ G, & F, & C, & N \end{matrix} \right\| = 0;$$

and not only so, but they touch at the points in question; so that, multiplying together the orders of the curve and surface, and subtracting *twice* the order of the point-system, we obtain the expression for the number of nodes. In the particular case where the functions are all linear, we have a sextic curve and cubic surface intersecting in 18 points; but the curve and surface touch in 4 points, and the number of nodes is $(18-2\,.\,4) = 10$. And in the same way the formula may be established for the general case.

8. The subsidiary theorem of the contact of the curve and surface requires, however, to be proved. Seeking for the equation of the tangent plane of the surface at any one of the points in question, we have first

$$\begin{vmatrix} \delta B, & \delta F, & \delta M \\ F, & C, & N \\ M, & N, & D \end{vmatrix} + \begin{vmatrix} B, & F, & M \\ \delta F, & \delta C, & \delta N \\ M, & N, & D \end{vmatrix} + \begin{vmatrix} B, & F, & M \\ F, & C, & N \\ \delta M, & \delta N, & \delta D \end{vmatrix} = 0,$$

tions represent curves having two common multiple points $\overline{xy}$, $\overline{xz}$ of the orders p, q, where $p+q=n$, we may write them in the forms

$$U = az^q + bxz^{q-1} + \&c.,$$
$$V = \alpha z^q + \beta xz^{q-1} + \&c.,$$

where a, b, &c., α, β, &c. are binary forms of the degree p in x, y.

The curve (2) now contains the line $x=0$; and if, as in the instance referred to, x is replaced by unity, this curve is lowered in degree by unity. The curve (1) is of the degree $2n-2$, and has $\overline{xy}$ for a multiple point of the order $2p-1$, and $\overline{xz}$ for a multiple point of the order $2q-2$; the curve (2) is to be taken as of the degree $2n-3$, with a multiple point at $\overline{xy}$ of the order $2p-2$, and another at $\overline{xz}$ of the order $2q-2$. There are also common tangents at the multiple points in number $2p-2$ and $2q-2$ respectively. The curves (3), (4) have the common multiple points $\overline{xy}$, $\overline{xz}$ of the orders p, $q-1$ respectively.

The order which we now obtain is

$$2(n-1)(2n-3)-2(p-1)(2p-1)-4(q-1)^2-2p-2q+4 \\ -\{(n-1)^2-p^2-(q-1)^2\},$$

or, what is the same thing, in accordance with Henrici's result,

$$6pq-4(p+q)+4.$$

5. It is easy to see geometrically that there is, in this point of view, a reduction by unity. For, taking a right line through the multiple points, we have n of the base-points linear; the remaining base-points lie, therefore, on a curve of the degree $n-1$ having multiple points at $\overline{xy}$, $\overline{xz}$ of the orders $p-1$, $q-1$ respectively. Hence the compound curve belongs to the involution, and the right line meets it again in one point, which constitutes a double point of the system.

If one of the multiple points is of the order $n-1$, every transversal through it meets the curve again in a single point, and the sum of the multiplicities is n. When, therefore, we discard the composite curves, there is no double point distinct from the given multiple point, as is otherwise evident.

If, again, we have any number m of multiple points on a transversal, the sum of whose multiplicities is n, it follows from the same reasoning that one of the curves of the involution contains the right line. When, therefore, we discard this case, which gives rise to $m-1$ other double points, the order of the discriminant of the corresponding form is

$$3(n-1)^2-\Sigma_m(p_1-1)(3p_1+1)-m+1 = 6\Sigma_m p_1 p_2-4\Sigma_m p_1+4.$$

The President gave an account of his investigations on the centro-surface of an ellipsoid (locus of the centres of curvature of the ellipsoid). The surface has been studied by Dr. Salmon, and also by Prof. Clebsch, but in particular the theory of the nodal curve on the surface admits of further development. The position of a point on the ellipsoid is de-

termined by means of the parameters, or elliptic coordinates, h, k; viz., if as usual a, b, c are the semi-axes, and if (X, Y, Z) are the coordinates of the point in question,

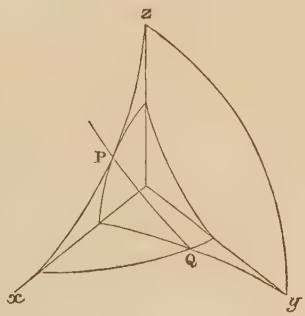
then
$$\frac{X^2}{a^2+h} + \frac{Y^2}{b^2+h} + \frac{Z^2}{c^2+h} = 1,$$
$$\frac{X^2}{a^2+k} + \frac{Y^2}{b^2+k} + \frac{Z^2}{c^2+k} = 1;$$

and hence
$$-\beta\gamma X^2 = a^2(a^2+h)(a^2+k),$$
$$-\gamma\alpha Y^2 = b^2(b^2+h)(b^2+k),$$
$$-\alpha\beta Z^2 = c^2(c^2+h)(c^2+k),$$

if for shortness
$$\alpha = b^2-c^2,$$
$$\beta = c^2-a^2,$$
$$\gamma = a^2-b^2,$$
$$(\alpha+\beta+\gamma = 0).$$

This being so, the coordinates of the point of intersection of the normal at (X, Y, Z) by the normal at the consecutive point of the curve of curvature
$$\frac{X^2}{a^2+k} + \frac{Y^2}{b^2+k} + \frac{Z^2}{c^2+k} = 1$$
are given by the formulæ
$$-\beta\gamma a^2x^2 = (a^2+h)^3(a^2+k),$$
$$-\gamma\alpha b^2y^2 = (b^2+h)^3(b^2+k),$$
$$-\alpha\beta c^2z^2 = (c^2+h)^3(c^2+k);$$

viz., these equations, considering therein (h, k) as arbitrary parameters, determine the coordinates (x, y, z) of a point on the centro-surface. The principal sections (as is known) consist each of them of an ellipse counting three times, and of an evolute of an ellipse; the evolute and ellipse have four contacts (two-fold intersections) and four simple intersections, but the contacts and intersections respectively, are in the different sections real and imaginary; and if (as, without loss of generality, may be assumed) $a^2+c^2 > 2b^2$, then the form of the principal sections is as shown in the figure (which represents only an octant of the surface); viz., there is a real contact at P in the plane of xz, and a real intersection at Q in the plane of xy. The surface has thus an exterior and an interior sheet, but (instead of meeting in a conical point, as in the wave surface) these intersect in a nodal curve QP. (The curve has a cusp at Q, and a node at P; viz., the curve extends beyond P, but from that point is acnodal, or without any real sheet of the



surface passing through it.) For the nodal curve there must be two values (h, k), (h_1, k_1), giving the same values of (x, y, z); viz., there must exist the relations

$$(a^2+h)^3(a^2+k) = (a^2+h_1)^3(a^2+k_1),$$
$$(b^2+h)^3(b^2+k) = (b^2+h_1)^3(b^2+k_1),$$
$$(c^2+h)^3(c^2+k) = (c^2+h_1)^3(c^2+k_1);$$

from which equations eliminating h_1 and k_1, we should have between h, k a relation which, combined with the expressions of x, y, z in terms of (h, k), determines the nodal curve. But the better course is to eliminate k, k_1, thus obtaining a relation between h and h_1, in virtue whereof h_1 may be regarded as a known function of h; k and k_1 can then be readily expressed in terms of h, h_1; that is, we have k as a function of h, h_1, or in effect as a function of h. The relation between h, h_1 (after a reduction of some complexity) assumes ultimately a form which is very simple and remarkable; viz., writing

$$P = a^2+b^2+c^2, \quad Q = b^2c^2+c^2a^2+a^2b^2, \quad R = a^2b^2c^2,$$

the relation is

$$(6R+3Qh+Ph^2),$$
$$+h_1(3Q+4Ph+3h^2),$$
$$+h_1^2(P \quad +3h \quad) = 0;$$

this is a $(2, 2)$ correspondence between the two parameters h, h_1; the united values $h_1 = h$, are given by the equation $6(R+Qh+Ph^2+h^3) = 0$, that is

$$(a^2+h)(b^2+h)(c^2+h) = 0;$$

viz., the two points on the ellipsoid which have their common centre of curvature on the nodal curve are only situate on the same curve of curvature when this curve is a principal section of the ellipsoid.

[Since the date of the foregoing communication, Prof. Cayley has found that the squared coordinates x^2, y^2, z^2 of a point on the nodal curve can be expressed as rational functions of a single variable parameter σ.]

The following present was received:—

"Monatsbericht" (Sept. and Oct. 1869).

January 13th, 1870.

Prof. CAYLEY, President, in the Chair.

Prof. Oppermann, of Copenhagen, was one of the visitors present.

Mr. A. Ramsay was proposed for election.

Mr. J. J. Walker, M.A., gave an account of his paper on the "Equations of Centres and Foci, and conditions of certain Involutions."

Dr. Henrici, Prof. Hirst, the President, and Mr. Clifford, took part in a discussion upon the paper.

The President then made his first communication to the Society of results he had arrived at in his paper on "Quartic Surfaces."

Mr. S. Roberts exhibited and explained diagrams of the Pedals of Conic Sections which he had constructed by the methods described in his communication of the 14th Jan. 1869 (vol. ii. p. 125).

The following presents were received:—

"A Memoir on Cubic Surfaces, and a Memoir on the Theory of Reciprocal Surfaces," by Prof. Cayley: from the Author.

"Annali di Matematica:" (Dicembre, 1869).

"Crelle's Journal," 71 Band, zweites Heft.

"Note on a Theorem relative to Neutral Series," by A. De Morgan, F.R.A.S., (Cambridge Phil. Society's Transactions, vol. xi. pt. 3): from the Author.

"On Interpolation, Summation, and the Adjustment of Numerical Tables," (Journal of Institute of Actuaries, vol. xi. p. 61); and "On an Improved Theory of Annuities and Assurances," (ditto vol. xv. p. 95,) from the Author, W. S. B. Woolhouse, F.R.A.S.

"On the Nodal Cones of Quadrinodal Cubics, and the Zomal Conics of Tetrazomal Quartics," by the Rev. R. Townsend, F.R.S.: from the Author.

"Monatsbericht" (Nov. 1869).

February 10*th*, 1870.

Prof. CAYLEY, President, in the Chair.

Prof. Oppermann, of Copenhagen, was again present.

Mr. A. Ramsay was elected a Member; and Mr. E. A. L. Bradshaw Smith, M.A., Fellow of Christ's College, Cambridge, was proposed for election.

The President having vacated the Chair, communicated the concluding portion of

A Memoir on Quartic Surfaces.

The present Memoir is intended as a commencement of the theory of the quartic surfaces which have nodes (conical points). A quartic surface may be without nodes, or it may have any number of nodes up to 16. I show that this is so, and I consider how many of the nodes may be given points. Although it would at first sight appear that the number is 8, it is in fact 7; viz., we can, with 7 given points as nodes

(but not in a proper sense with 8 or more given points), find a quartic surface; such surface contains in its equation 6 constants, which may be such that the surface has an additional node or nodes. Suppose that the surface has an 8th node:—there are two distinct cases; viz., (1) the 8 nodes are the points of intersection of 3 quadric surfaces, or say they are an octad, and the surface is said to be octadic; (2) the 8th node is any point whatever on a certain sextic surface determined by means of the 7 given nodes, and called the dianodal surface of these 7 points; the quartic surface is said to be a dianome. The two cases are in general exclusive of each other; viz., the 7 given points being any points whatever, the dianodal surface does not pass through the 8th point of the octad; so that the quartic surface with the 8 nodes is either octadic or else a dianome. Assuming it to be a dianome, the constants may be further determined so that there shall be a 9th node; it is necessary to examine whether this forms with 7 of the 8 nodes an octad. Supposing that it does not (viz., that there are not any 8 nodes in regard to which the surface is octadic), the 9th node is then any point whatever on a certain curve of the order 18, determined by means of the 8 nodes, and called the dianodal curve of these 8 points. And, finally, the constants may be further determined so that there shall be a 10th node; supposing, as before, that this does not form an octad with any 7 of the 9 nodes (viz., that there are not any 8 nodes in regard to which the surface is octadic), the 10th node is then any one of a system of 22 points determined by means of the 9 nodes, and called the dianodal system of these 9 points. But the quartic surface is now completely determined; viz., starting with any 7 given points as nodes, we have a dianome with 8 nodes, 9 nodes, or 10 nodes, say, an octodianome, enneadianome, or decadianome, but not with any greater number of nodes; these can only present themselves when particular conditions are satisfied in regard to the 7 given nodes, and to the 8th and 9th node; and the consideration of the quartic surfaces with more than 10 nodes would thus form a separate branch of the subject.

The case of the decadianome (or quartic surface with 10 nodes formed as above with 7 given points as nodes) is peculiarly interesting. I identify this with the surface which I call a symmetroid; viz., the surface represented by an equation $\Delta = 0$, where Δ is a *symmetrical* determinant of the 4th order the several terms whereof are linear functions of the coordinates (x, y, z, w); this surface is related to the Jacobian surface of 4 quadric surfaces (itself a very remarkable surface), and this theory of the symmetroid and the Jacobian, and of questions connected therewith, forms a large portion of the present Memoir.

The theory of the Jacobian is connected also with the researches in regard to nodal quartic surfaces in general; and, for greater clearness, it

has seemed to me proper to commence the Memoir with certain definitions, &c., in regard to this theory. It will be seen in what manner I extend the notion of the Jacobian.

I remark that the present researches on Quartic Surfaces were suggested to me by Professor Kummer's most interesting Memoir "Ueber die Algebraischen Strahlensysteme &c.," Berl. Abh., 1866, in which, without entering upon the general theory, he is led to consider the quartic surfaces, or certain quartic surfaces, with 16, 15, 14, 13, 12, or 11 nodes; the last of these, or surface with 11 nodes, being in fact a particular case of the symmetroid.

Considerations in regard to the Jacobian of four, or more or less than four Surfaces.

1. In the case of any four surfaces, $P=0$, $Q=0$, $R=0$, $S=0$, the differential coefficients of P, Q, R, S in regard to the coordinates (x, y, z, w) may be arranged as a square matrix in either of the ways

$$\begin{array}{c|c} & \underline{P,\ Q,\ R,\ S} \\ \delta_x & \\ \delta_y & \\ \delta_z & \\ \delta_w & \end{array} \quad ; \quad \begin{array}{c|c} \underline{\delta_x,\ \delta_y,\ \delta_z,\ \delta_w} & \\ P & \\ Q & \\ R & \\ S & \end{array}$$

and with either arrangement we may form one and the same determinant, the Jacobian determinant J (P, Q, R, S), or, equating it to zero, the Jacobian surface J (P, Q, R, S) = 0, of the four surfaces.

2. In the case of more than four surfaces, adopting the arrangement

$$\begin{array}{c|c} & \underline{P,\ Q,\ R,\ S,\ T\ \cdots} \\ \delta_x & \\ \delta_y & \\ \delta_z & \\ \delta_w & \end{array}$$

and considering the several determinants which can be formed with any four columns of the matrix, these equated to zero establish a more than one-fold relation between the coordinates; viz., in the case of five surfaces, we have $J(P, Q, R, S, T) \equiv 0$, a two-fold relation representing a curve; and in the case of six surfaces, $J(P, Q, R, S, T, U) \equiv 0$, a three-fold relation representing a point-system; and (since with four coordinates a relation is at most three-fold) these are the only cases to be considered.

3. In the case of fewer than four surfaces, adopting the arrangement

$$\begin{array}{c|c} \underline{\delta_x,\ \delta_y,\ \delta_z,\ \delta_w} & \\ P & \\ Q & \\ \vdots & \end{array}$$

and considering the several determinants which can be formed with any 3 or 2 columns of the matrix, and equating these to zero, we have in like manner a more than one-fold relation between the coordinates; viz., in the case of three surfaces, we have $J(P, Q, R) \equiv 0$, a two-fold relation representing a curve; and in the case of two surfaces $J(P, Q) \equiv 0$, a three-fold equation representing a point-system, (viz., this denotes the points $\delta_x P : \delta_y P : \delta_z P : \delta_w P = \delta_x Q : \delta_y Q : \delta_z Q : \delta_w Q$); for a single surface we should have a four-fold relation, and the case is not considered. But observe that if the notation were used, $J(P) \equiv 0$ would denote $\delta_x P = 0$, $\delta_y P = 0$, $\delta_z P = 0$, $\delta_w P = 0$, equations which are satisfied simultaneously by the coordinates (x, y, z, w) of any node of the surface $P = 0$. Although in what precedes I have used the sign $\equiv$, there is no objection to using, and I shall in the sequel use, the ordinary sign $=$, it being understood that while $J(P, Q, R, S) = 0$ denotes a single equation or one-fold relation, $J(P, Q, R, S, T) = 0$ or $J(P, Q, R) = 0$ will each denote a two-fold relation, and $J(P, Q, R, S, T, U) = 0$ or $J(P, Q) = 0$ each of them a three-fold relation.

4. It is not asserted that ... $J(P, Q, R) = 0$, $J(P, Q, R, S) = 0$, $J(P, Q, R, S, T) = 0$, ... form a continuous series of analogous relations; and there might even be a propriety in using, in regard to four or more surfaces, J, and in regard to four or fewer surfaces an inverted J (viz., in regard to four surfaces, either symbol indifferently); but there is no ambiguity in, and I have preferred to adopt the use of, the single symbol J.

5. Suppose that the orders of the surfaces $P = 0$, $Q = 0$, ... are $a+1$, $b+1$, ... (so that the orders of the differential coefficients of P, Q, ... are $a, b, \ldots$), then we have for the orders of the several loci,

$J(P, Q) = 0$,	point-system,	order	$a^3+a^2b+ab^2+b^3$;
$J(P, Q, R) = 0$,	curve,	„	$a^2+b^2+c^2+bc+ca+ab$;
$J(P, Q, R, S) = 0$,	surface,	„	$a+b+c+d$;
$J(P, Q, R, S, T) = 0$,	curve,	„	$ab+ac \ldots +de$;
$J(P, Q, R, S, T, U) = 0$,	point-system,	„	$abc+abd \ldots +def$.

See, as to this, Salmon's "Solid Geometry," Ed. 2, (1865), "On the Order of Systems of Equations." In particular, if $a = b = c \ldots = 1$, then the orders are 4, 6, 4, 10, 20.

As to the Surface obtained by equating to zero a Symmetrical Determinant.

6. It is also shown (Salmon, p. 495) that the surface obtained by equating to zero any symmetrical determinant has a determinate number of nodes; viz., if the orders of the terms in the diagonal be

a, b, c, &c., then the number of nodes is $= \frac{1}{2}(\Sigma a \,.\, \Sigma ab - \Sigma abc)$, or, as this may also be written, $\frac{1}{2}(\Sigma a^2 b + 2\Sigma abc)$. In particular, the formula applies to the case of the surface

$$\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix} = 0,$$

(a, b, c, d) being here the orders of A, B, C, D respectively, and the orders of F, G, &c., being $\frac{1}{2}(b+c)$, $\frac{1}{2}(a+c)$, &c. If the terms are all of them linear functions of the coordinates, or $a=b=c=d=1$, then the number of nodes is $=10$.

7. That the surface has nodes is, in fact, clear from the consideration that any point for which the minors of the determinant all vanish will be a node; and that (for the symmetrical determinant), by making the minors all of them vanish, we establish only a three-fold relation between the coordinates. The expression for the number of the nodes is, I think, obtained most readily as follows:

The nodes will be points of intersection of the curve and surface

$$\left\| \begin{matrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \end{matrix} \right\| = 0, \quad \begin{vmatrix} B, & F, & M \\ F, & C, & N \\ M, & N, & D \end{vmatrix} = 0,$$

these, however, contain in common the points

$$\left\| \begin{matrix} H, & B, & F, & M \\ G, & F, & C, & N \end{matrix} \right\| = 0;$$

and not only so, but they touch at the points in question; so that, multiplying together the orders of the curve and surface, and subtracting *twice* the order of the point-system, we obtain the expression for the number of nodes. In the particular case where the functions are all linear, we have a sextic curve and cubic surface intersecting in 18 points; but the curve and surface touch in 4 points, and the number of nodes is $(18-2\,.\,4) = 10$. And in the same way the formula may be established for the general case.

8. The subsidiary theorem of the contact of the curve and surface requires, however, to be proved. Seeking for the equation of the tangent plane of the surface at any one of the points in question, we have first

$$\begin{vmatrix} \delta B, & \delta F, & \delta M \\ F, & C, & N \\ M, & N, & D \end{vmatrix} + \begin{vmatrix} B, & F, & M \\ \delta F, & \delta C, & \delta N \\ M, & N, & D \end{vmatrix} + \begin{vmatrix} B, & F, & M \\ F, & C, & N \\ \delta M, & \delta N, & \delta D \end{vmatrix} = 0,$$

where, in virtue of the equations

$$\left\| \begin{matrix} H, & B, & F, & M \\ G, & F, & C, & N \end{matrix} \right\| = 0,$$

the last term vanishes. Expanding the other two terms, the equation becomes

$$D(C\delta B + B\delta C - 2F\delta F) - (N^2\delta B - 2MN\delta F + M^2\delta C) + \delta M(FN - CM) + \delta N(BN - MF) = 0;$$

but, in virtue of the same equations, the coefficients of δM and δN each of them vanish, and we have also

$$N^2\delta B + M^2\delta C - 2MN\delta F = \frac{N^2}{C}(C\delta B + B\delta C - 2F\delta F);$$

so that the equation becomes finally $C\delta B + B\delta C - 2F\delta F = 0$. Investigating by a like process the equation of the tangent of the curve

$$\left\| \begin{matrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C & N \end{matrix} \right\| = 0,$$

we find between the differentials δA, δB, &c., a two-fold linear relation, expressible by means of the foregoing equation $C\delta B + B\delta C - 2F\delta F = 0$, and one other equation; that is, at each of the points in question the tangent of the curve lies in the tangent plane of the surface, or, what is the same thing, the curve and surface touch at these points.

Surfaces represented by an equation F (P, Q) = 0, &c.

9. In the remarks which follow as to the surfaces F (P, Q) = 0, F (P, Q, R) = 0, &c., the function F is a rational and integral function of (P, Q), (P, Q, R), &c., not in general homogeneous in regard to P, Q, R, ... but of such degrees in regard to these functions respectively as to be homogeneous in regard to the coordinates (x, y, z, w).

The surface F (P, Q) = 0 has in general a nodal curve $\delta_P F = 0$, $\delta_Q F = 0$; and if it has besides any nodes, these are points of the point-system J (P, Q) = 0.

The surface F (P, Q, R) = 0 has in general nodes $\delta_P F = 0$, $\delta_Q F = 0$, $\delta_R F = 0$; and if it has besides any nodes, these are points on the curve J (P, Q, R) = 0.

The surface F (P, Q, R, S) = 0 has not in general, but it may have, nodes $\delta_P F = 0$, $\delta_Q F = 0$, $\delta_R F = 0$, $\delta_S F = 0$; if it has any other nodes, these are points on the surface J (P, Q, R, S) = 0.

Nodes of a Quartic Surface; Circumscribed Cone having its vertex at a Node.

10. A quartic surface may be without nodes; or it may have any number of nodes up to 16. Consider a quartic surface having a node

or nodes; and take the single node, or (if more nodes than one) any one of the nodes, as the vertex of a circumscribed cone; then, considering any plane through the vertex, the section will be a quartic curve having a node at the vertex, and the generating lines in the plane will be the tangents from the node to the quartic curve; the number of them is therefore 6, and the order of the circumscribed cone is thus $= 6$. Each tangent intersects the quartic curve in the node counting as two intersections, and in the point of contact counting as two intersections; there are consequently no singular tangents; and therefore in the circumscribed cone no singular lines arising from a singular tangency of the generating line. Hence, in the case of a single node on the surface, the circumscribed cone is a cone of the order 6 without nodal or stationary lines; and the class is $= 30$. But in the case of more than one node, say k nodes, the circumscribed cone passes through the remaining $(k-1)$ nodes, and the generating line through each of these nodes is a nodal line of the cone; that is, the cone has $(k-1)$ nodal lines, and its class is $= 30-2k+2$. The cone is not of necessity a proper cone; the maximum number of nodal lines is when it breaks up into 6 planes, and we have then $k-1=15$; that is, the number of nodes of the surface is at most $= 16$.

11. It is easy to form a table of the different *primâ facie* possible forms of the sextic cone, according to the number of nodes of the surface; viz., writing 6 for a proper sextic cone without nodal lines, $6_1, 6_2 \ldots 6_{10}$ for the proper sextic cone with 1, 2, ... or 10 nodal lines; and so $5, 5_1 \ldots 5_6$ for the proper quintic cones, $4, 4_1, 4_2, 4_3, 3, 3_1, 2$ for the quartic, cubic, and quadric cones, and 1 for the plane, the table is

Nodes of Surface.	Circumscribed Sextic Cone.							
1	6							
2	6_1							
3	6_2							
4	6_3							
5	6_4							
6	6_5;	$5,\ 1$						
7	6_6;	$5_1, 1$						
8	6_7;	$5_2, 1$						
9	6_8;	$5_3, 1$;	$4,\ 2$					
10	6_9;	$5_4, 1$;	$4_1, 2$;	$4,\ 1, 1$;	$3,\ 3$			
11	6_{10};	$5_5, 1$;	$4_2, 2$;	$4_1, 1, 1$;	$3_1, 3$			
12	...	$5_6, 1$;	$4_3, 2$;	$4_2, 1, 1$;	$3_1, 3_1$;	$3,\ 2, 1$		
13	...	...	...	$4_3, 1, 1$;	...	$3_1, 2, 1$;	$3,\ 1, 1, 1$;	$2,\ 2, 2$
14	...	...	...	...	...	...	$3_1, 1, 1, 1$;	$2,\ 2, 1, 1$
15	...	...	...	...	...	...	...	$2_1, 1, 1, 1, 1$
16	...	...	...	...	...	...	...	$1,\ 1, 1, 1, 1, 1$

And moreover, in the cases where there are two or more forms of the sextic cone, then the k sextic cones may be of the different forms in various combinations. The total number of cases *primâ facie* possible is thus very great; but only a comparatively small number of them actually exist.

12. In the case where there is a plane 1, the sextic cone breaks up into this plane, and into a (proper or improper) quintic cone intersecting the plane in 5 lines; that is, there will be in the plane 6 nodes; the plane is, in fact, a singular tangent plane meeting the surface in a conic twice repeated; and the 6 nodes lie on this conic. Taking any one of these nodes as vertex, the corresponding sextic cone breaks up into the plane, and into a (proper or improper) quintic cone.

13. In the cases $k = 1, 2, 3, 4, 5$, and $k = 15, 16$, there is only one form of sextic cone; so that each node (at least so far as appears) stands in the same relation to the surface. Considering the last mentioned two cases; $k = 16$,—each of the 16 nodes gives 6 singular tangent planes, but each of these passes through 6 nodes; therefore the number of planes is $= 16$: similarly, $k = 15$, the number of singular tangent planes is $15 \times 4 \div 6, = 10$.

For $k = 14$, the cones are $3_1, 1, 1, 1$, or $2, 2, 1, 1$: it is easy to see that we have only the three cases

Cones $3_1, 1, 1, 1 : 2, 2, 1, 1$

N° may be	14	, 0	gives	$(14.3 + 0.2) \div 6, = 7$	singular
„	8	, 6	„	$(8.3 + 6.2) \div 6, = 6$	tangent
„	2	, 12	„	$(2.3 + 12.2) \div 6, = 5$	planes;

and we may in the like manner limit the number of possible cases, for other values of k. But I do not at present further pursue the inquiry.

As to the Number of Constants contained in a Surface.

14. We say that a surface $P = 0$ contains or depends upon a certain number of constants; viz., this is the number of constants contained in the equation $P = 0$ of the surface, taking the coefficient of any one term to be equal to unity; thus the general quadric surface contains 9 constants; the surface can in fact be determined so as to satisfy 9 conditions; or, as we might express it, the *postulation* of the surface is $= 9$. And if, in the general equation so containing 9 constants, k of these are given, or, what is the same thing, if the quadric surface be made to satisfy any k conditions, then the number of constants, or postulation of the surface, is $= 9 - k$.

15. But a different form of expression is sometimes convenient: the conditions to be satisfied are frequently such that, being satisfied by

the surfaces $P = 0$, $Q = 0$, ..., they will be satisfied by the surface $\alpha P + \beta Q + \ldots = 0$, where $\alpha, \beta, \ldots$ are any constant multipliers whatever. When this is so, there will be a certain number of solutions $P = 0$, $Q = 0, \ldots$ not connected by any such relation, or say of asyzygetic solutions, such that the general surface satisfying the conditions in question is $\alpha P + \beta Q + \ldots = 0$; and hence, taking one of these coefficients as unity, the number of constants, or postulation of the surface, is equal to the number of the remaining coefficients, or, what is the same thing, it is less by unity than the number of the asyzygetic solutions $P = 0$, $Q = 0, \ldots$ Instead of considering the number of constants, or postulation, we may consider the number of solutions (that is, asyzygetic solutions) or surfaces $P = 0$, $Q = 0, \ldots$ which satisfy the conditions in question.

16. Thus, for the quadric not subjected to any conditions, there are 10 surfaces (for example, these may be taken to be the surfaces $x^2 = 0$, $y^2 = 0$, $z^2 = 0$, $w^2 = 0$, $yz = 0$, $zx = 0$, $xy = 0$, $xw = 0$, $yw = 0$, $zw = 0$); and the general quadric surface is by means of these expressed linearly in the form $(a, \ldots \mathcal{X} x, y, z, w)^2 = 0$. So for the quadric surfaces through k given points, the number of these is $= 10 - k$; thus for the surfaces through 4 given points, say the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, the 6 given surfaces may be taken to be $yz = 0$, $zx = 0$, $xy = 0$, $xw = 0$, $yw = 0$, $zw = 0$, and every other quadric surface through the 4 points is by means of these expressed linearly in the form $(a, \ldots \mathcal{X} yz, zx, xy, xw, yw, zw) = 0$; for the quadric surfaces through 8 points there are two surfaces $P = 0$, $Q = 0$; and every quadric surface through the 8 points is by means of these expressed linearly in the form $\alpha P + \beta Q = 0$; and (as the extreme case) if the quadric surface passes through 9 given points, then there is the single quadric surface $P = 0$.

17. In the questions in which such quadric surfaces present themselves, it is in general quite immaterial what particular surfaces are selected as the surfaces $P = 0$, $Q = 0$, ...; the selection may be made at pleasure and, being so made, the surfaces are to be regarded as completely determinate; viz., there would be no gain of generality if these were replaced by any other surfaces $\alpha P + \beta Q \ldots = 0$. For instance, in the theory of the quartic surfaces with 6 given points as nodes, we have through the 6 given points the 4 quartic surfaces $P = 0$, $Q = 0$, $R = 0$, $S = 0$, and we consider the quartic functions $(a, \ldots \mathcal{X} P, Q, R, S)^2$ and $J(P, Q, R, S)$: each of these is unaltered as to its form when P, Q, R, S are replaced each of them by any linear function of these quantities; viz., $(a, \ldots \mathcal{X} P, Q, R, S)^2$ is changed into a new quadric function $(a', \ldots \mathcal{X} P, Q, R, S)^2$, and $J(P, Q, R, S)$ into a mere constant multiple of its original value. We have herein a justification of the expressions in question, through 6 given points there are 4 quadric surfaces, &c.

General theory of the Quartic Surface with a given Node or Nodes.

18. A quartic surface contains 34 constants; and the number of conditions to be satisfied in order that a given point may be a node is $= 4$. Hence, if the surface has k given points as nodes, the number of constants is $= 34-4k$; and it would at first sight appear that k might be $= 8$, and that with the 8 given points as nodes we should have a quartic surface containing 2 constants. But this is not so in a proper sense; for through the 8 given points we have 2 quadric surfaces $P = 0$, $Q = 0$; and we can by means of these form a quartic surface $(a, b, c\,\text{X}\,P, Q)^2 = 0$, containing 2 constants, and having in a sense the 8 points as nodes. This, however, is no proper quartic surface, but is a system of 2 quadric surfaces, each of them passing through the 8 points, and the two quadric surfaces therefore intersecting in a quadriquadric curve through the 8 points; which curve is therefore a nodal curve on the compound surface; and it is only as points on this nodal curve, and not in a proper sense, that the 8 given points are nodes of the quartic surface. The greatest value of k is thus $k = 7$.

19. Of course, if $k = 0$, we have the general quartic surface $U = 0$, containing 34 constants. The cases $k = 1$, $k = 2$, $k = 3$ (viz., a single given node, 2 given nodes, 3 given nodes), may be at once disposed of; taking for instance the 1st node to be the point $(1, 0, 0, 0)$, the 2nd node the point $(0, 1, 0, 0)$, the 3rd node the point $(0, 0, 1, 0)$, we find at once an equation $U=0$, with 30, 26, or 22 constants, having the given node or nodes.

Four given Nodes.

20. The case of 4 given nodes is just as easy; but in reference to what follows, it is proper to consider it more in detail. The equation should contain 18 constants; we have through the 4 given points 6 quadric surfaces, $P=0$, $Q=0$, $R=0$, $S=0$, $T=0$, $U=0$, and we can by means of them form a quartic equation $(a,\ldots\,\text{X}\,P, Q, R, S, T, U)^2= 0$ having the 4 given points as nodes; this contains, however, $(21-1 =)$ 20 constants; the reduction to the right number 18 occurs by reason that the functions (P, Q, R, S, T, U), although linearly independent, are connected by two quadric equations

$$(*\,\text{X}\,P, Q, R, S, T, U)^2 = 0, \quad (*'\,\text{X}\,P, Q, R, S, T, U)^2 = 0;$$

hence writing the equation of the quartic surface in the form

$$(a, \ldots\,\text{X}_{,,})^2 - \lambda\,(*\,\text{X}_{,,})^2 - \mu\,(*'\,\text{X}_{,,})^2 = 0,$$

the coefficients λ, μ may be so determined as to reduce to zero the coefficients of any two terms of the equation, and the number of constants really is $20-2 = 18$, as it should be.

21. In proof, observe that, taking the 4 given nodes to be the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(1, 0, 0, 0)$, the quadric surfaces may

be taken to be $yz = 0$, $zx = 0$, $xy = 0$, $xw = 0$, $yw = 0$, $zw = 0$; the equation of the quartic surface will thus be

$$(a, \ldots \text{\textbackslash\!\!\!\textbackslash}yz, zx, xy, xw, yw, zw)^2 = 0;$$

but we have between the functions xy, &c., the two identical relations

$$xy \,.\, zw - xz \,.\, yw = 0, \quad xy \,.\, zw - xw \,.\, yz = 0;$$

and the number of constants is thus $= 18$.

Five given Nodes.

22. In the case of 5 given nodes, the number of constants should be $= 14$. We have through the 5 given points, 5 quadric surfaces $P = 0$, $Q = 0$, $R = 0$, $S = 0$, $T = 0$, and we form herewith the quartic equation $(a, \ldots \text{\textbackslash\!\!\!\textbackslash}P, Q, R, S, T)^2 = 0$, containing the right number 14 of arbitrary constants. The functions P, Q, &c. are in this case not connected by any quadric relation, and the equation just written down is in fact the general equation of the quartic surface with the 5 given nodes.

23. In verification, take the first 4 nodes to be as above, and the 5th node to be the point (1, 1, 1, 1); we may write

$$(P, Q, R, S, T) = \{x(y-z),\ x(y-w),\ y(x-z),\ y(x-w),\ xy-zw\};$$

and if from the 5 equations $P = x(y-z)$, &c., we eliminate (x, y, z, w), we obtain one, and only one, relation between the functions P, Q, R, S, T; this is found to be

$$PS(Q+R-T) - QR(P+S-T) = 0,$$

or, what is the same thing,

$$R(P-Q)(S-T) - P(R-S)(Q-T) = 0;$$

viz., it is a cubic relation, and there is consequently no quadric relation between the 5 functions.

Six given Nodes.

24. In the case of 6 given nodes, the quartic surface should contain 10 constants. We have through the 6 given points 4 quadric surfaces $P = 0$, $Q = 0$, $R = 0$, $S = 0$; but if we form herewith the quartic surface $(a, \ldots \text{\textbackslash\!\!\!\textbackslash}P, Q, R, S)^2 = 0$, this contains only 9 constants. It is to be shown that the Jacobian surface $J(P, Q, R, S) = 0$ of the 4 quadric surfaces (or say of the 6 points) is a quartic surface having the 6 given points as nodes, and not included in the foregoing form $(a, \ldots \text{\textbackslash\!\!\!\textbackslash}P, Q, R, S)^2 = 0$; this being so, we have the quartic surface

$$(a, \ldots \text{\textbackslash\!\!\!\textbackslash}P, Q, R, S)^2 + \theta J(P, Q, R, S) = 0,$$

having the 6 given points as nodes, and containing the complete number of constants, viz. 10.

25. The 6 given nodes being any points whatever, their coordinates may be taken to be (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1),

$(1, 1, 1, 1)$, and $(\alpha, \beta, \gamma, \delta)$. I proceed to find the Jacobian of these 6 points. For this purpose, let (a, b, c, f, g, h) be the 6 coordinates of the line through the points $(1, 1, 1, 1)$ and $(\alpha, \beta, \gamma, \delta)$, viz.,

$$\begin{aligned} a &= \beta - \gamma, & f &= \alpha - \delta, \\ b &= \gamma - \alpha, & g &= \beta - \delta, \\ c &= \alpha - \beta, & h &= \gamma - \delta, \end{aligned}$$

and consequently,

$$\begin{aligned} . \quad h - g + a &= 0, \\ -h \quad . + f + b &= 0, \\ g - f \quad . + c &= 0, \\ -a - b - c \quad . &= 0, \\ af + bg + ch &= 0, \text{ \&c.;} \end{aligned}$$

we have through the 6 points the plane pairs

$$\begin{aligned} x\,(\quad . \quad hx - gz + aw) &= 0, \\ y\,(-hx \quad . + fz + bw) &= 0, \\ z\,(\quad gx - fy \quad . + cw) &= 0, \\ w\,(-ax - by - cz \quad .\,) &= 0, \end{aligned}$$

where, adding the four equations, we have identically $0 = 0$. For this reason, we cannot take these to be the equations of the 4 quadric surfaces, but we may take the first 3 of them for the surfaces $P = 0$, $Q = 0$, $R = 0$; and for the 4th surface $S = 0$, I take the quadric cone having its vertex at the point $(0, 0, 0, 1)$; viz., the equation is

$$a\alpha yz + b\beta zx + c\gamma xy = 0;$$

that is, I write

$$(P, Q, R, S) = \{x(hy - gz + aw),\ y(-hx + fz + bw),\ z(gx - fy + cw),\ (a\alpha yz + b\beta zx + c\gamma xy)\}.$$

26. The Jacobian is then easily found to be

$$\begin{aligned} &(b\beta zx + c\gamma xy)\ (-agh,\ bhf,\ cfg,\ abc,\ -af^2,\ -g\mathrm{B},\ h\mathrm{C},\ a\mathrm{A},\ b^2g,\ -c^2h \,\mathrm{\Sigma}\, x, y, z, w)^2 \\ +\,&(c\gamma xy + a\alpha yz)\ (agh,\ -bhf,\ cfg,\ abc,\ f\mathrm{A},\ -bg^2,\ -h\mathrm{C},\ -a^2f,\ b\mathrm{B},\ c^2h \,\mathrm{\Sigma}\, x, y, z, w)^2 \\ +\,&(a\alpha yz + b\beta zx)\ (agh,\ bhf,\ -cfg,\ abc,\ -f\mathrm{A},\ g\mathrm{B},\ -ch^2,\ a^2f,\ -b^2g,\ c\mathrm{C} \,\mathrm{\Sigma}\, x, y, z, w)^2 \\ &\qquad\qquad = 0; \end{aligned}$$

where for the moment A, B, C denote $bg - ch$, $ch - af$, $af - bg$ respectively. Collecting and reducing, the whole divides by $2abc$; and if finally we replace a, b, c, f, g, h by their values, the result is

$$\mathrm{J} = \left\{ \begin{aligned} &(\beta - \gamma)\ yz\ (\alpha w^2 - \delta x^2) + (\alpha - \delta)\ xw\ (\beta z^2 - \gamma y^2) \\ +\,&(\gamma - \alpha)\ zx\ (\beta w^2 - \delta y^2) + (\beta - \delta)\ yw\ (\gamma x^2 - \alpha z^2) \\ +\,&(\alpha - \beta)\ xy\ (\gamma w^2 - \delta z^2) + (\gamma - \delta)\ zw\ (\alpha y^2 - \beta x^2) \end{aligned} \right\} = 0.$$

27. It may be shown *à posteriori* that J is not a quadric function of P, Q, R, S. For, attempting to express it in this form, J does not contain the terms x^2w^2, y^2w^2, z^2w^2, and it thence at once appears that the coefficients of P^2, Q^2, R^2 each of them vanish. Hence, introducing for convenience

the factor 2, I assume $(0, 0, 0, D, F, G, H, L, M, N \chi P, Q, R, S)^2 = 2J$. Comparing the terms in w^2 (yz, zx, xy), we obtain

$$bcF = a\alpha, \quad caG = b\beta, \quad abH = c\gamma;$$

and comparing the coefficients of w $(y^2z, z^2x, x^2y, yz^2, zx^2, xy^2)$, we obtain

$$-Ff + a\alpha M = \frac{h\alpha}{b}, \quad Ff + a\alpha N = -\frac{g\alpha}{c},$$

$$-Gg + b\beta N = \frac{f\beta}{c}, \quad Gg + b\beta L = -\frac{h\beta}{a},$$

$$-Hh + c\gamma L = \frac{g\gamma}{a}, \quad Hh + c\gamma M = -\frac{f\gamma}{b};$$

substituting for F, G, H their values, we obtain from the first 3 equations $L, M, N = \frac{-f}{bc}, \frac{-g}{ca}, \frac{-h}{ab}$, and from the second 3 equations, $L, M, N = \frac{f}{bc}, \frac{g}{ca}, \frac{h}{ab}$; that is, the equations are inconsistent, and the function J is not expressible in the form in question.

Jacobian Surface of Six given Points.

28. The equation $J = 0$ is the locus of the vertices of the quadric cones which pass through the given 6 points; calling these 1, 2, 3, 4, 5, 6, we see at once that the surface passes through the 15 lines 12, 13, ... 56, and also through the ten lines 123 . 456 (viz., line of intersection of the planes through 1, 2, 3, and through 4, 5, 6), &c. In fact, taking the vertex at any point O in the line 1, 2, the lines drawn to the six points are O1 = O2, O3, O4, O5, O6; viz., there are only five lines, so that these lie in a quadric cone. And taking the vertex at any point in the line 123 . 456, the lines to the 6 points lie in these planes 123 and 456 respectively, and the quadric cone is in fact this plane-pair. Moreover, the surface containing the lines 12, 13, 14, 15, 16, must have the point 1 for a node; and similarly, the points 2, 3, 4, 5, 6 are each of them a node on the surface. It is to be added that the surface contains the skew cubic through the 6 points, or say the skew cubic 123456. See, as to this, *post* No. 108.

29. The surface in question (the Jacobian of the 6 points) is a particular case of the Jacobian of any 4 quadric surfaces. This more general surface will be considered in the sequel; I only remark here that it contains 10 lines, corresponding to the 10 lines 123 . 456, &c., but it has not any other lines, or any nodes.

Jacobian Curve of Seven given Points, or of an Octad of Points.

30. In connexion with what precedes, we may here consider a curve which presents itself in the sequel; viz., the curve which is the locus of

the vertices of the quadric cones which pass through seven given points. The general case is when no one of the points is the vertex of a quadric cone through the other 6 points. We have through the 7 points the three quadric surfaces $P = 0$, $Q = 0$, $R = 0$; hence, forming the equation $\alpha P + \beta Q + \gamma R = 0$ of the general quadric surface through the 7 points, and making this a cone, we find as the locus of the vertex $J(P, Q, R) = 0$; the analytical form shows that this is a sextic curve. It appears, moreover, that the curve is symmetrically related to all the 8 points $P = 0$, $Q = 0$, $R = 0$; and instead of calling it the Jacobian of the 7 points, we may call it the Jacobian of the octad. But in further explanation, take the points to be 1, 2, 3, 4, 5, 6, 7; the vertex will lie on each of the Jacobian surfaces 123456 and 123457; and it is at present assumed that 7 is not a point on the first surface, nor 6 a point on the second surface. The two surfaces have in common the lines 12, 13, ... 45, and they consequently besides intersect in a curve of the 6th order, or sextic curve, which is the locus in question. At the point 1 there is on the first surface a tangent cone through the lines 12, 13, 14, 15, 16, and on the second surface a tangent cone through the lines 12, 13, 14, 15, 17; these two cones have for their complete intersection the lines 12, 13, 14, 15, which lines belong to the complete intersection of the two surfaces, but not to the sextic curve. It thus appears, *à posteriori*, that the sextic curve does not pass through the point 1; and similarly, that it does not pass through any of the points 2, 3, 4, or 5. As to the points 6 and 7, each of these is on only one of the quartic surfaces, and therefore the curve of intersection does not pass through either of these points.

31. Suppose, however, that one of the seven points is the vertex of a cone through the other six; it is of course the same thing whether we take this to be one of the points 1, 2, 3, 4, 5, or one of the points 6 and 7, but the result comes out more easily in the latter case; viz., in the former case, taking 1 to be the point in question, the two tangent cones at 1 are one and the same cone, and all that appears is that there is nothing to hinder a branch or branches of the sextic curve from passing through the point 1. But in the latter case, taking 7 for the point in question, then 7 lies on the surface 123456, being a simple point on this surface, but a node on the surface 123457; and it thus appears that there are through 7 two branches of the sextic curve; so that any one of the seven points, being the vertex of a cone through the other six, is an actual double point on the sextic curve.

32. In the case where two of the points are each of them the vertex of a cone through the other six points, then the seven points lie on a skew cubic; and the sextic curve of the general case becomes this skew cubic twice repeated.

Seven given Nodes.

33. In the case of 7 given nodes, the number of constants should be $= 6$; the 7 given points determine 3 quadric surfaces $P = 0$, $Q = 0$, $R = 0$; and we have hence the quartic surface $(a, \ldots \lambda P, Q, R)^2 = 0$, containing 5 constants only. That this is not the general quartic surface with the 7 given nodes, is also clear from the consideration that the surface in question has 8 nodes; viz., the 8 points of intersection of the three quadric surfaces. Suppose that a particular quartic surface, having the 7 given nodes, but not of the last mentioned form, is $\Delta = 0$; then a quartic surface having the 7 given nodes is

$$(a, \ldots \lambda P, Q, R)^2 + \theta\Delta = 0;$$

and this, as containing 6 constants, will be the general quartic surface with the 7 given nodes.

34. It follows that, if $\Delta' = 0$ be another quartic surface having the 7 given nodes, we must have identically $\Delta' - p\Delta = (* \lambda P, Q, R)^2$, where p is a determinate constant and $(* \lambda P, Q, R)^2$ a determinate quadric function of (P, Q, R). The formula extends to the case where $\Delta' = 0$ has the 8 nodes $(P = 0, Q = 0, R = 0)$, but we have then $p = 0$, and the meaning is simply that the general quartic surface having the 8 nodes is $(* \lambda P, Q, R)^2 = 0$.

35. A particular quartic surface having (in an improper sense) the 7 given nodes, but not having the 8th node, is $M\Omega = 0$, where $M = 0$ in the plane through any 3 of the 7 points and $\Omega = 0$ is the cubic surface through these same 3 points, and having the remaining 4 points as nodes. The equation of the cubic surface, if the 4 points are taken to be $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, is obviously of the form

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} + \frac{d}{w} = 0, \quad (\text{that is, } ayzw + bzxw + cxyw + dxyz = 0),$$

and by making the surface pass through the 3 points we determine linearly the coefficients (a, b, c, d), that is, their ratios. The equation of the quartic surface thus is

$$(a, \ldots \lambda P, Q, R)^2 + \theta M\Omega = 0,$$

the 7 given points being here proper nodes; and the formula being precisely equivalent to the preceding one containing Δ.

36. We can with the 7 given points form 35 such combinations $M\Omega = 0$ of a plane and a cubic surface, and so present the equation of the quartic surface under 35 different forms; these are of course equivalent in virtue of the before mentioned formula for $\Delta' - p\Delta$; viz., we must have identically $M\Omega - pM'\Omega' = (* \lambda P, Q, R)^2$: a theorem of some interest, which it might be difficult to verify *à posteriori*.

Investigation of the cases of 8 *Nodes.*

37. It has been already shown that a quartic surface cannot in a proper sense have 8 given nodes. In regard to the quartic surfaces with 8 nodes, we start from the surface with 7 given nodes; viz.,

$$(a, \ldots \mathcal{J}\mathrm{P}, \mathrm{Q}, \mathrm{R})^2 + \theta\nabla = 0,$$

or what is the same thing

$$(a, \ldots \mathcal{J}\mathrm{P}, \mathrm{Q}, \mathrm{R})^2 + \theta\mathrm{M}\Omega = 0;$$

and we inquire in what cases this surface has an 8th node. Obviously if $\theta = 0$, that is, if the surface is $(a, \ldots \mathcal{J}\mathrm{P}, \mathrm{Q}, \mathrm{R})^2 = 0$, the surface will have an 8th node, the remaining intersection of the quadric surfaces $\mathrm{P} = 0$, $\mathrm{Q} = 0$, $\mathrm{R} = 0$ (observe that this is a point in no wise depending on the particular quadric surfaces, but uniquely determined by means of the 7 given points); and we have thus one kind, say the "octadic" surface, of the quartic surfaces with 8 nodes; viz., the nodes are the 8 points of intersection of any 3 quadric surfaces, or they are an octad of points. By what precedes, 7 of the nodes may be given points, and the remaining node is then a uniquely determinate point, the 8th point of the octad.

38. But if θ be not $= 0$, there may still be an 8th node; viz., this must then be a point on the Jacobian surface $\mathrm{J}(\mathrm{P}, \mathrm{Q}, \mathrm{R}, \nabla) = 0$, which is of the order 6. It is clear *à priori* that this must be a surface depending only on the 7 points, but independent of the particular surfaces $\mathrm{P} = 0$, $\mathrm{Q} = 0$, $\mathrm{R} = 0$, $\nabla = 0$; to verify this, observe that, substituting for ∇ the function $\nabla' = p\nabla + (* \mathcal{J}\mathrm{P}, \mathrm{Q}, \mathrm{R})^2$, we in fact leave the Jacobian unaltered; I call it the dianodal surface of the 7 points.

39. I say that the 8th node may be any point whatever on the dianodal surface; in fact, regarding for a moment the coordinates of the node as given, and expressing that the point is a node on the quartic surface, we have 4 equations containing

$$a\mathrm{P}_0 + h\mathrm{Q}_0 + g\mathrm{R}_0,\ h\mathrm{P}_0 + b\mathrm{Q}_0 + f\mathrm{R}_0,\ g\mathrm{P}_0 + f\mathrm{Q}_0 + c\mathrm{R}_0,$$

(P_0, Q_0, R_0 the values of P, Q, R at the node,) but which, if only the point be on the dianodal surface, reduce themselves to three equations; viz., we have between the coefficients (a, b, c, f, g, h) and θ three equations which being satisfied, the point in question will be a node. And it thus appears that, taking the 8th node to be a given point on the dianodal surface, the equation $(a, \ldots \mathcal{J}\mathrm{P}, \mathrm{Q}, \mathrm{R})^2 + \theta\nabla = 0$ of the quartic surface will contain 3 constants. Observe that we may through the 8 nodes draw 2 quadric surfaces $\mathrm{P} = 0$, $\mathrm{Q} = 0$; and this being so if $\Delta = 0$ be a particular quartic surface with the 8 nodes, then the general quartic surface will be $(a, b, c \mathcal{J}\mathrm{P}, \mathrm{Q})^2 + \theta\Delta = 0$,

containing the right number 3 of constants. But there is not here any simple form of the surface $\Delta = 0$, such as the form $M\Omega = 0$ for the surface through 7 given points.

40. It is clear *à priori* that the relation between the 8 nodes is a symmetrical one; so that the 8th point being situate anywhere on the dianodal surface of the 7 points, each of the points will be situate on the dianodal surface of the remaining 7 points. This is a remarkable property of the dianodal surface, which will have to be again considered.

41. In what precedes, we have the second kind of quartic surfaces with 8 nodes, say the "dianome"; viz., each node is a point on the dianodal surface of the remaining 7 nodes; any 7 of the nodes may be taken to be given points, and the remaining node to be any point whatever on the dianodal surface of the 7 points.

The Dianodal Surface.

42. Consider the seven points 1, 2, 3, 4, 5, 6, 7. As already mentioned, through three of these, say 1, 2, 3, we may draw a plane $M=0$; and through the same three points, with the remaining points 4, 5, 6, 7 as nodes ($3+4.4=19$ conditions), a cubic surface $\Omega=0$; this surface passing through the six lines 45, 46, ... 67. Hence we have $\Delta, = M\Omega, = 0$, a quartic surface with the seven points as nodes. And using this form of Δ, it may be shown that the dianodal $J(P, Q, R, \Delta) = 0$ passes through the 21 lines 12, 13, ... 67, and through 35 plane cubics such as $M=0$, $\Omega=0$; viz., this is a cubic in the plane 123 passing through the points 1, 2, 3, and through the intersections of the plane with each of the six lines 45, 46, ... 67 (nine points determining the cubic); the complete intersection by the plane 123 being therefore composed of this cubic and of the three lines 12, 13, 23. For the passage through the cubic, we have only to observe that

$$J(P, Q, R, M\Omega) = J(P, Q, R, \Omega)\, M + J(P, Q, R, M)\, \Omega = 0$$

is satisfied by $M=0$, $\Omega=0$; and for the passage through the lines, taking $x=0$, $y=0$, $z=0$, $w=0$ for the equations of the planes 567, 674, 745, and 456 respectively, each of the functions P, Q, R is of the form $ayz+bzx+cxy+fxw+gyw+hzw$, and the function Ω is of the form $Ayzw+Bzwx+Cwxy+Dxyz$. Hence, writing in the derived functions for instance $z=0$, $w=0$, the first and second lines of the determinant $J(P, Q, R, \Omega)$ will be of the form

$$\begin{vmatrix} cy, & c'y, & c''y, & 0 \\ cx, & c'x, & c''x, & 0 \end{vmatrix},$$

or the determinant vanishes for $z=0$, $w=0$; that is, for any point of the line 45 we have $\Omega=0$ and also $J(P, Q, R, \Omega) = 0$; consequently

$J(P, Q, R, M\Omega) = 0$, and the like for the other lines. The theorem is thus proved.

43. I say that the dianodal surface passes through each of the 7 skew cubics, such as 123456. To prove this, it is only necessary to show that the skew cubic 123456 lies on the dianodal surface. For this purpose it will be enough to show that the skew cubic meets the plane 712 in a point of the surface; for then it will, in like manner, meet each of the 15 planes 712, 713, ... 756 in a point of the surface; that is, we shall have 15 intersections of the curve and surface, and there are, besides, the intersections 1, 2, 3, 4, 5, 6, in all 21 intersections; that is, the skew cubic must lie on the surface.

44. The plane 712 meets the surface in three lines and in a plane cubic determined by the points 7, 1, 2 and the six intersections of the plane with the lines 34, 35, ... 56. We have therefore to show that this plane cubic meets the skew cubic 123456. Consider for a moment the points 1, 2, 3, 4, 5, 6 and another point 7′. As seen above, we have in general, through the points 1, 2, 7′ and with the points 3, 4, 5, 6 as nodes, a determinate cubic surface, which surface passes through the lines 34, 35, ... 56. But the cubic surface becomes indeterminate if the points 1, 2, 7′, 3, 4, 5, 6 are on the same skew cubic; that is, if 7′ is any point whatever on the skew cubic 123456 (the proof presently). Taking, then, 7′ as the intersection of the skew cubic by the plane 712, we have in this plane the points 7′, 1, 2, and the intersections of the plane by the lines 34, 35, ... 56, nine points through which there pass an infinity of plane cubics; that is, the plane cubic determined by the points 7, 1, 2 and the six intersections will pass through the point 7′; viz., it meets the skew cubic 123456.

45. For the subsidiary theorem, taking X, Y, Z, W as current coordinates, viz., $X=0$, $Y=0$, $Z=0$, $W=0$ as the equations of the planes 456, 563, 634, 345 respectively, (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) as the coordinates of the points 1 and 2 respectively, and (x, y, z, w) for those of 7′; the equation of the cubic surface passing through 7′, 1, 2, and having the nodes 3, 4, 5, 6, is

$$\begin{vmatrix} \frac{1}{X}, & \frac{1}{Y}, & \frac{1}{Z}, & \frac{1}{W} \\ \frac{1}{x}, & \frac{1}{y}, & \frac{1}{z}, & \frac{1}{w} \\ \frac{1}{x_1}, & \frac{1}{y_1}, & \frac{1}{z_1}, & \frac{1}{w_1} \\ \frac{1}{x_2}, & \frac{1}{y_2}, & \frac{1}{z_2}, & \frac{1}{w_2} \end{vmatrix} = 0;$$

and this ceases to be a determinate function if only

$$\left\| \begin{matrix} \frac{1}{x}, & \frac{1}{y}, & \frac{1}{z}, & \frac{1}{w} \\ \frac{1}{x_1}, & \frac{1}{y_1}, & \frac{1}{z_1}, & \frac{1}{w_1} \\ \frac{1}{x_2}, & \frac{1}{y_2}, & \frac{1}{z_2}, & \frac{1}{w_2} \end{matrix} \right\| = 0;$$

viz., considering (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) as given, this is a two-fold relation between the coordinates (x, y, z, w) of the point 7′. The relation may be represented by the four equations $(yzw) = 0$, $(zwx) = 0$, $(wxy) = 0$, $(xyz) = 0$, if for shortness

$$(yzw) = \begin{vmatrix} yz, & zw, & wy \\ y_1z_1, & z_1w_1, & w_1y_1 \\ y_2z_2, & z_2w_2, & w_2y_2 \end{vmatrix}$$

and the like as to the other symbols. The four equations represent quadric surfaces, each two intersecting in a line [*e. g.*, $(yzw) = 0$, $(zwx) = 0$ in the line $z = 0$, $w = 0$], and the four surfaces besides intersecting in a skew cubic, which is the required locus of the point 7′, and which, as is seen at once, passes through the points 1, 2, 3, 4, 5, 6.

46. By what precedes, we have on the dianodal surface through the point 1 the lines 12, 13, 14, 15, 16, 17, and the skew cubics 123456, &c. The six lines are not on the same quadric cone, and it thus appears that the point 1 must be a cubic-node (point where, instead of the tangent plane, we have a cubic cone) on the surface. It is to be remarked that the lines 12, 13, 14, 15, 16, and the tangent at 1 to the skew cubic 123456, lie in a quadric cone; viz., this tangent is given as the sixth intersection of the cubic cone with the quadric cone through the lines 12, 13, 14, 15, 16.

47. I revert to the equation of the dianodal surface as given in the form $J = J(P, Q, R, M\Omega) = 0$, where $M = 0$ is the plane through the points 1, 2, 3, and $\Omega = 0$ the cubic surface through these points, and having the points 4, 5, 6, 7, as nodes. We can find the orders of the several functions P, Q, R, M, Ω in the coordinates (x_1, y_1, z_1, w_1), &c., of the several points; viz., writing for shortness x_1^2 to denote the order 2 in regard to (x_1, y_1, z_1, w_1), and so in other cases, we have

$$\begin{aligned} P = Q = R &= x^2 (x_5, x_6, x_7)^2 (x_1, x_2, x_3, x_4)^2 \\ M &= x (x_5, x_6, x_7). \\ \Omega &= x^3 (x_5, x_6, x_7)^3 (x_1, x_2, x_3, x_4)^3; \end{aligned}$$

[where, of course, the x^2, x, x^3 show in like manner the orders in regard to the current co-ordinates (x, y, z, w); the proof in regard to

Ω is easily supplied.] The order of J is equal that of PQRMΩ, less 4 as regards the current coordinates, by reason of the differentiations; that is, we have $J = x^6 (x_1 x_2 x_3)^{10} (x_4 x_5 x_6 x_7)^{15}$; and we thus see that the equation of the dianodal surface as above obtained is encumbered with a constant factor of the form $(x_1 x_2 x_3)^4 (x_4 x_5 x_6 x_7)^9$. In fact, the relation between the 7 points and the current point (x, y, z, w), or say the point 8, as expressing that the 8 points are the nodes of a dianome, should be a symmetrical one in regard to the coordinates of the several points; and being of the order 6 in regard to the coordinates (x, y, z, w), it should be of the same order in regard to the other coordinates; that is, the true form would be $J = (x\, x_1 x_2 x_3 x_4 x_5 x_6 x_7)^6 = 0$.

48. It is possible that taking the 4 points, say 1, 2, 3, 4, to be $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$, and the 3 points, say 5, 6, 7, to be $(1, 1, 1, 1), (\alpha, \beta, \gamma, \delta), (\alpha', \beta', \gamma', \delta')$, the extraneous factor might exhibit itself, and that the equation divested of this factor might be of a tolerably simple form. I have not, however, worked this out, but I have, by an independent process, obtained in regard to the dianodal surface of the 7 points a result which may be interesting.

49. The dianodal surface, *quà* surface having the first-mentioned 4 points for cubic nodes, has its equation of the form

$$yzw\,(y, z, w)^3 + zxw\,(z, x, w)^3 + xyw\,(x, y, w)^3 + xyz\,(x, y, z)^3 + xyzw\,(x, y, z, w)^2 = 0;$$

where in the cubic functions the terms x^3, y^3, z^3, w^3 none of them appear. If for instance $w=0$, the equation becomes $(x, y, z)^3 = 0$, which, by what precedes, is a known cubic curve, viz. the curve through the points 1, 2, 3 and the intersections of the plane 123 by the lines 45, 46, 47, 56, 57, 67; and we can by this consideration find the cubic function $(x, y, z)^3$, and thence by symmetry the other cubic functions. I take

$$\left.\begin{matrix}(a, b, c, f, g, h)\\ (a', b', c', f', g', h')\\ (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{f}, \mathrm{g}, \mathrm{h})\end{matrix}\right\} \text{ for coordinates of line through } \left\{\begin{matrix}(1, 1, 1, 1),\ (\alpha, \beta, \gamma, \delta)\\ (1, 1, 1, 1),\ (\alpha', \beta', \gamma', \delta')\\ (\alpha, \beta, \gamma, \delta),\ (\alpha', \beta', \gamma', \delta')\end{matrix}\right.$$

respectively; viz., I write

$$\begin{array}{l|l|l} a=\beta-\gamma,\ f=\alpha-\delta & a'=\beta'-\gamma',\ f'=\alpha'-\delta' & \mathrm{a}=\beta\gamma'-\beta'\gamma,\ \mathrm{f}=\alpha\delta'-\alpha'\delta \\ b=\gamma-\alpha,\ g=\beta-\delta & b'=\gamma'-\alpha',\ g'=\beta'-\delta' & \mathrm{b}=\gamma\alpha'-\gamma'\alpha,\ \mathrm{g}=\beta\delta'-\beta'\delta \\ c=\alpha-\beta,\ h=\gamma-\delta & c'=\alpha'-\beta',\ h'=\gamma'-\delta' & \mathrm{c}=\alpha\beta'-\alpha'\beta,\ \mathrm{h}=\gamma\delta'-\gamma'\delta \end{array}$$

and I write moreover

$$\begin{aligned} \lambda &= \ .\ \mathrm{h}-\mathrm{g}+\mathrm{a}, \\ \mu &= -\mathrm{h}\ .+\mathrm{f}+\mathrm{b}, \\ \nu &= \ \mathrm{g}-\mathrm{f}\ .+\mathrm{c}, \\ \varpi &= -\mathrm{a}-\mathrm{b}-\mathrm{c}\ . \end{aligned}$$

50. This being so, the cubic curve through the last mentioned six points has its equation of the form

$$\frac{A}{ax+by+cz}+\frac{B}{a'x+b'y+c'z}+\frac{C}{\mathrm{a}x+\mathrm{h}y+\mathrm{g}z}+\frac{D}{\lambda x+\mu y+\nu z}=0;$$

and to make this pass through the points 1, 2, 3, we write therein successively $(y=0, z=0)$, $(z=0, x=0)$, $(x=0, y=0)$; viz., we have for the ratios A : B : C : D the three equations

$$\frac{A}{a}+\frac{B}{a'}+\frac{C}{\mathrm{a}}+\frac{D}{\lambda}=0,$$
$$\frac{A}{b}+\frac{B}{b'}+\frac{C}{\mathrm{h}}+\frac{D}{\mu}=0,$$
$$\frac{A}{c}+\frac{B}{c'}+\frac{C}{\mathrm{g}}+\frac{D}{\nu}=0.$$

In eliminating, for instance, B for the first and second equations, the resulting equation divides by $ab'-a'b$, $\doteq \mathrm{a}+\mathrm{b}+\mathrm{c}$, and we thus obtain, between A, C, D, the three equations (equivalent to two)

$$\frac{A}{bc}+\frac{Ca'}{\mathrm{bc}}+\frac{Df'}{\mu\nu}=0,$$
$$\frac{A}{ca}+\frac{C\beta'}{\mathrm{ca}}+\frac{Dg'}{\nu\lambda}=0,$$
$$\frac{A}{ab}+\frac{C\gamma'}{\mathrm{ab}}+\frac{Dh'}{\lambda\mu}=0,$$

from which the ratios A : C : D may be obtained by actual calculation. After all reductions, we have

$$\mathrm{A}= \quad abc\ \{(a'\delta'+\beta'\gamma')\,\mathrm{af}+(\beta'\delta'+\gamma'a')\,\mathrm{bg}+(\gamma'\delta'+a'\beta')\,\mathrm{ch}\},$$
$$\mathrm{B}=-a'b'c'\ \{(a\delta+\beta\gamma)\,\mathrm{af}+(\beta\delta+\gamma a)\,\mathrm{bg}+(\gamma\delta+a\beta)\,\mathrm{ch}\},$$
$$\mathrm{C}= \quad \mathrm{abc}\ \{aa'\lambda+\beta\beta'\mu+\gamma\gamma'\nu+\delta\delta'\varpi\},$$
$$\mathrm{D}=-\lambda\mu\nu\ \{aa'\mathrm{a}+\beta\beta'\mathrm{b}+\gamma\gamma'\mathrm{c}\};$$

viz., A, B, C, D are proportional to these values respectively. Multiplying by the product of the denominators, I find without much difficulty that the resulting cubic function is divisible by $\mathrm{a}+\mathrm{b}+\mathrm{c}$; hence, introducing the factor xyz, and an indeterminate multiplier l, I write

$$xyz\,(x,y,z)^3=$$
$$\frac{l}{\mathrm{a}+\mathrm{b}+\mathrm{c}}xyz\,(ax+by+cz)\,(a'x+b'y+c'z)\,(\mathrm{a}x+\mathrm{b}y+\mathrm{c}z)\,(\lambda x+\mu y+\nu z)$$
$$\times\left\{\frac{A}{ax+by+cz}+\frac{B}{a'x+b'y+c'z}+\frac{C}{\mathrm{a}x+\mathrm{b}y+\mathrm{c}z}+\frac{D}{\lambda x+\mu y+\nu z}\right\},$$

where A, B, C, D have the values above written down.

51. Considering the orders in regard to $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$, and observing that a, b, c and a', b', c' are linear functions of the two sets

respectively, but that a, b...h, $\lambda...\varpi$, are linear in the two sets conjointly, or say $\alpha, .. = \alpha,\ \alpha', ... = \alpha'$; $a, ... = a\alpha'$;
we have $$A\alpha'a\lambda = \alpha^5\alpha'^4 . \alpha^2\alpha'^3 = \alpha^7\alpha'^7,$$
so that after the division by $a+b+c, = a\alpha'$, the order will be $\alpha^6\alpha'^6$. Hence l will be a mere numerical factor, and the last-mentioned equation gives, without any extraneous factor, the terms $xyz\ (x, y, z,)^3$ in the equation of the dianodal surface of the seven points.

Octadic Surfaces with 9 *or* 10 *Nodes.*

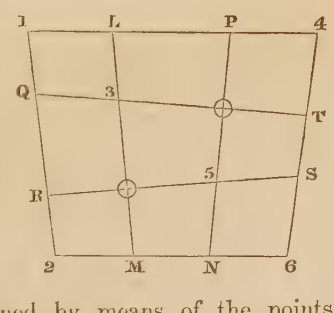
52. In regard to the surfaces with 9 and 10 nodes, I consider first the octadic surfaces. Starting as before with the given points 1, 2, 3, 4, 5, 6, 7, we have a determinate point 8 completing the octad, and the surface with the 8 nodes is
$$(\alpha, ...\text{ }\text{X}\text{ P, Q, R})^2 = 0,$$
(5 constants). Suppose that there is another node 9; this must be a point on the Jacobian curve J (P, Q, R) = 0, which (as was seen) is a sextic curve not passing through any of the 8 points; the node 9 may be *any* point on this curve, viz., taking its coordinates as given, the condition of its being a node gives 4 equations, which, for the very reason that the point is on the Jacobian curve, reduce themselves to 2 equations, which can be satisfied by means of the constants $(\alpha, ...)$, the resulting equation should therefore contain 3 constants.

53. In order to find it, taking as above 9 a given point on the Jacobian curve, this will be the vertex of a quadric cone, say P=0, through the 8 points; we may draw through the 9 points another quadric surface Q=0, and through the 8 points a quadric surface R=0; this being so, we have the quartic surface $(a, b, 0, 0, g, h\text{ X P, Q, R})^2 = 0$, having the 9 nodes, and containing, as it should do, 3 constants; this may be written $$(a\text{P}+2h\text{Q}+2g\text{R})\text{ P}+b\text{Q}^2 = 0;$$
viz., if $b\text{R}' = a\text{P}+2h\text{Q}+2g\text{R}$, that is, if R′=0 be the general quadric surface through the 8 points, then the equation is $\text{Q}^2-\text{PR}' = 0$, where observe that R′ is considered as containing implicitly 3 constants.

54. If there is a 10th node, say 10, this is also a point on the Jacobian curve J (P, Q, R) = 0, and it may be any point whatever on the curve; taking it as a given point on the curve, the resulting equation should contain 1 constant. We may take P=0 to be the quadric cone, vertex 9, through the 8 points, R=0 the quadric cone, vertex 10, through the 8 points, Q=0 the quadric surface through the 8 points and the points 9 and 10 (viz., the surface through 9, 10 and any 7 of the 8 points will pass through the remaining 8th point). The equation of the quartic surface then is
$$(0, b, 0, 0, g, 0\text{ X P, Q, R})^2 = 0;$$

that is, $bQ^2+2gPR=0$, containing 1 constant; we may reduce this to $Q^2-PR=0$, the constant being considered as contained implicitly in one of the functions. It is clear that the constant cannot be so determined as to give rise to an 11th node, nor indeed to any other singularity in the surface.

55. In the case of the surface with 9 nodes, it is clear that this is octadic in one way only; the node 9 cannot form an octad with any 7 of the remaining nodes. But in the case of the surface with 10 nodes, the question arises whether the nodes 9 and 10 may not be such as to form an octad with some six, say with the nodes 1, 2, 3, 4, 5, 6 of the remaining 8 nodes; that is, whether we can have 1, 2, 3, 4, 5, 6, 7, 8 forming an octad, and also 1, 2, 3, 4, 5, 6, 9, 10 forming an octad. I will show that this is impossible if only the points 1, 2, 3, 4, 5, 6 are given points, that is, points assumed at pleasure and not specially related to each other. For this purpose, assuming that the points form 2 octads as above, take through 1, 2, 3, 4, 5, 6, 7, 9 the quadric surfaces $P=0$, $Q=0$, then each of these passes through 8, 10; take $R=0$ any other quadric surface through 1, 2, 3, 4, 5, 6, 7, 8, and $S=0$ any other quadric surface through 1, 2, 3, 4, 5, 6, 9, 10. Then $P=0$, $Q=0$, $R=0$ intersect in the 1st octad, and $P=0$, $Q=0$, $S=0$ intersect in the 2nd octad; the quartic surface (if it exists) must be simultaneously of the forms $(*\,\mathfrak{J}P, Q, R)^2=0$, $(*\,\mathfrak{J}P, Q, S)^2=0$; and this implies an identical equation $(*\,\mathfrak{J}P, Q, R, S)^2=0$. The quadric surfaces are surfaces through the points 1, 2, 3, 4, 5, 6, and taking through these six points any other quadric surfaces $A=0$, $C=0$, $E=0$, $H=0$, we have P, Q, R, S each of them a linear function of A, C, E, H; and the relation between P, Q, R, S gives a like relation $(*\,\mathfrak{J}A, C, E, H)^2=0$ between A, C, E, H. I assume $A=123\,.\,456$, $E=134\,.\,256$, $H=145\,.\,236$, $C=152\,.\,346$; viz., $A=0$ is the plane-pair formed of the planes through 1, 2, 3 and 4, 5, 6 respectively; and so for the others: we have to show that there is not any such identical relation $(*\,\mathfrak{J}A, C, E, H)^2=0$.



56. We may through 3 draw the lines LM, QT to meet 14, 26 and 12, 46 respectively; and through 5 the lines RS, NP to meet 14, 26 and 12, 46 respectively. Observe that the points O in the figure are apparent intersections only; viz., NP does *not* meet QT, *nor* LM meet RS. In fact, if NP met QT it would be a line in the series of lines meeting 14, QT, 26; or 5 would be situate in a hyperboloid, determined by means of the points

1, 2, 4, 6, 3; viz., 5 would not be an arbitrary point: and so LM does not meet RS. Now the quadrics E, H meet in the lines 14, 26, LM, NP, and the quadrics A, C in the lines 12, 46, QT, RS. Suppose that we had identically $(*\,\square\,\mathrm{A, C, E, H})^2 = 0$; putting therein E=0, H=0, we should have $(*\,\square\,\mathrm{A, C})^2 = 0$, viz., $(\mathrm{A}+\lambda\mathrm{C})(\mathrm{A}+\mu\mathrm{C}) = 0$; or there would exist quadrics of the forms $\mathrm{A}+\lambda\mathrm{C} = 0$ containing the lines 14, 26, LM, NP. Now there is no quadric surface $\mathrm{A}+\lambda\mathrm{C} = 0$ containing the line NP; for $\mathrm{A}+\lambda\mathrm{C} = 0$ is a quadric containing the sides of the quadrilateral QRST; the generating lines of the one kind meet each of the lines RS, QT; those of the other kind neither. Hence NP, which meets RS but not QT, cannot be a generating line of either kind; and we have no identical relation $(\mathrm{A, C, E, H})^2 = 0$.

57. In the octadic surface with 9 nodes; starting with any 7 nodes of the octad, 9 is not the 8th point of the octad, and hence (by the theory of the dianome) it must lie in the dianodal surface of the 7 points; that is, the dianodal surface of the 7 points must pass through 9, viz., through any point whatever of the Jacobian curve of the 7 points, that is, of the octad; or (what is the same thing) the dianodal surface of the 7 points passes through the Jacobian curve of the octad. This is an obvious property of the dianodal surface, the surface $\mathrm{J}(\mathrm{P, Q, R}, \nabla) = 0$ contains the Jacobian curve $\mathrm{J}(\mathrm{P, Q, R}) = 0$. But it further appears that, starting with any 6 points of the octad and with the point 9 (that is, any point whatever of the Jacobian curve), the dianodal surface of these 7 points must contain the remaining 2 points of the octad. And in the octadic surface with 10 nodes, starting with any 5 points of the octad and with the points 9 and 10 (that is, any two points on the Jacobian curve) the dianodal surface of these 7 points must contain the remaining three points of the octad. I have not attempted to verify these last properties of the dianodal surface.

Dianomes with 9 *or* 10 *Nodes.*

58. I now consider the dianomes with 9 and 10 nodes. Starting from the general form

$$(a, b, c\,\square\,\mathrm{P, Q})^2 + \theta\Delta = 0,$$

where $\Delta = 0$ is a particular quartic surface having the 8 nodes, it at once appears that if there is a 9th node, say 9, this must be a point on the Jacobian curve $\mathrm{J}(\mathrm{P, Q}, \Delta) = 0$, or say on the dianodal curve of the 8 points, viz. ($a = b = 1$, $c = 3$, in the formula No. 5), this is a curve of the order 18; the node may be any point whatever on this curve, and taking it to be a given point on the curve, the number of constants in the resulting equation should be 1. Hence if $\mathrm{P} = 0$ be the quadric surface through the 9 points, and $\Delta = 0$ a particular quartic surface having the 9 points as nodes, the general equation is $a\mathrm{P}^2 + \theta\Delta = 0$.

59. But we may consider the question somewhat differently. Starting with the 7 given points 1, 2, 3, 4, 5, 6, 7 and with 8 a given point on the dianodal surface of the 7 points; it is clear that 9 must be on the dianodal surface 1234567, and also on the dianodal surface 1234568; the complete intersection is of the order 36, and we have to consider how this breaks up so as to contain as part of itself the dianodal curve of the order 18.

Dianodal Curve of 8 *Points.*

60. Consider first any 8 points whatever 1, 2, 3, 4, 5, 6, 7, 8; where 8 is not on the dianodal surface 1234567, nor 7 on the dianodal surface 1234568. The two surfaces have in common the 15 lines 12, 13, ... 56 and the skew cubic 123456, they therefore besides intersect in a curve of the order 18. At the point 1 the tangent cubic cones of the two surfaces intersect in the lines 12, 13, 14, 15, 16 and the tangent to the skew cubic 123456, 6 lines lying in a quadric cone; they therefore besides intersect in 3 lines lying in a plane; that is, the point 1 is on the curve of the order 18 an actual triple point, the 3 tangents lying *in plano;* and the like of course in regard to each of the points 2, 3, 4, 5, 6. But as 7, 8 lie each of them on only one of the two surfaces, the curve of the order 18 does not pass through 7 or 8.

61. If, however, 8 lies on the dianodal surface 1234567, then each of the 8 points will lie on the dianodal surface of the other 7; and in particular 7 will lie on the dianodal surface 1234568. The surfaces intersect as before in a residual curve of the order 18; the only difference is that 7 and 8 are now points on each surface; viz., each of them is on one of the surfaces an ordinary point, and on the other a cubic node; the points 7 and 8 are thus each of them an actual triple point on the curve; and at each of them the 3 tangents are *in plano.* We thus see that the dianodal curve 12345678 is a curve of the order 18, such that each of the 8 points is a triple point on the curve, the tangents at each of them being *in plano.*

Ten Nodes.

62. Suppose there is a 10th node, say 10; starting from the equation $a\mathrm{P}^2+\theta\Delta=0$ ($\mathrm{P}=0$ the quadric surface through the 9 points, $\Delta=0$ a particular quartic surface having the 9 points as nodes), it at once appears that the node must be one of the points $\mathrm{J}(\mathrm{P},\Delta)=0$; hence, taking it to be one of these points, we have 4 equations, which, in virtue of the node being one of the points in question, reduce themselves to a single equation determining the ratio $a:\theta$; we have thus a completely determinate surface, say $\square=0$ having the 10 points as nodes. The number of points $\mathrm{J}(\mathrm{P},\Delta)$, writing in the formula No. 5, $a=1$, $b=3$, is obtained as $1+3+9+27=40$, but it is to be observed that the

surface $P=0$ passes through each of the 9 nodes of the surface $\Delta=0$; these count twice among the points $J(P, \Delta) = 0$, and the number of residual points (or say the dianodal centres of the 9 points) is $40-18 = 22$; viz., this is the number of positions of the node 10.

Dianodal Centres of 9 Points.

63. In further explanation, observe that 9 is any point on the dianodal curve 12345678; the node 10 must lie on this same curve, and also on the dianodal surface 1234569. Take $P=0$ the quadric through all the 9 points, $Q = 0$ a quadric through all but the point 9, $R = 0$ through all but the point 8, $S = 0$ through all but the point 7. The dianodal curve 12345678 is $J(P, Q, \nabla) = 0$, and the dianodal surface 1234569 is $J(P, R, S, \nabla) = 0$; the total number of intersections is $6 \times 18 = 108$; these include the $4 \times 18 = 72$ points of intersection of the dianodal curve $J(P, Q, \Delta) = 0$ with the Jacobian surface $J(P, Q, R, S) = 0$, except the 4 points $J(P, Q) = 0$, which are the vertices of the 4 quadric cones through 1, 2, 3, 4, 5, 6, 7, 8 (which 4 points are not situate on the curve $J(P, R, S) = 0$), and there are besides 40 points $[108 = (72-4)+40]$ which are the before mentioned points $J(P, \Delta) = 0$; viz., these are the 9 points each twice, and the residual 22 points which are the dianodal centres of the 9 points.

General result as to the Dianomes.

64. We have thus established the theory of the dianome quartic surfaces; viz., we have

The octodianome, 8 nodes, 7 of them arbitrary, and the 8th an arbitrary point on the dianodal surface (order 6) of the 7 points.

The enneadianome, 9 nodes, the 9th an arbitrary point on the dianodal curve (order 18) of the 8 points.

The decadianome, 10 nodes, the 10th any one of the 22 dianodal centres of the 9 points.

And as already mentioned, so long as the first 7 nodes are arbitrary, there cannot be more than 10 nodes in all.

The Symmetroid.

The Lineolinear Correspondence of Quartic Surfaces.

65. I consider four equations $S=0$, $T=0$, $U=0$, $V=0$, lineolinear in regard to the two sets of coordinates (x, y, z, w) and $(\alpha, \beta, \gamma, \delta)$; viz., each of these equations is of the form

$$(* \; \mathfrak{x}\, x, y, z, w \, \mathfrak{x}\, \alpha, \beta, \gamma, \delta) = 0.$$

This implies that the point (x, y, z, w) lies on a certain quartic surface

$\Theta=0$, and the point $(\alpha, \beta, \gamma, \delta)$ on a certain quartic surface $\Delta=0$, and that the two surfaces correspond point to point to each other. In fact, writing the four equations in the form

$$\begin{aligned} L\alpha + M\beta + N\gamma + P\delta &= 0, \\ L'\alpha + M'\beta + N'\gamma + P'\delta &= 0, \\ L''\alpha + M''\beta + N''\gamma + P''\delta &= 0, \\ L'''\alpha + M'''\beta + N'''\gamma + P'''\delta &= 0, \end{aligned}$$

where L, &c., are linear functions of (x, y, z, w), then eliminating $(\alpha, \beta, \gamma, \delta)$, we obtain the equation

$$\Theta = \begin{vmatrix} L, & M, & N, & P \\ L', & M', & N', & P' \\ L'', & M'', & N'', & P'' \\ L''', & M''', & N''', & P''' \end{vmatrix} = 0;$$

and similarly, writing the four equations in the form

$$\begin{aligned} Ax + By + Cz + Dw &= 0, \\ A'x + B'y + C'z + D'w &= 0, \\ A''x + B''y + C''z + D''w &= 0, \\ A'''x + B'''y + C'''z + D'''w &= 0, \end{aligned}$$

where A, &c., are linear functions of $(\alpha, \beta, \gamma, \delta)$, then eliminating (x, y, z, w), we obtain the equation

$$\Delta = \begin{vmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{vmatrix} = 0.$$

Moreover, Θ being $=0$, the four linear equations in $(\alpha, \beta, \gamma, \delta)$ are equivalent to three equations, and give for instance $(\alpha, \beta, \gamma, \delta)$ proportional to the determinants formed with the matrix

$$\begin{vmatrix} L', & M', & N', & P' \\ L'', & M'', & N'', & P'' \\ L''', & M''', & N''', & P''' \end{vmatrix};$$

and similarly, Δ being $=0$, the four linear equations in (x, y, z, w) are equivalent to three equations, and give for instance (x, y, z, w) proportional to the determinants formed with the matrix

$$\begin{vmatrix} A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{vmatrix};$$

which establishes the point-to-point correspondence of the two surfaces.

66. It would at first sight appear that any quartic surface $(*\,\mathfrak{Q}\alpha, \beta, \gamma, \delta)^4 = 0$ whatever might have its equation expressed in the foregoing determinant form $\Delta = 0$. This equation seems, in fact,

to contain homogeneously as many as 64 constants. But if we multiply the determinant line into line by a constant determinant

$$\begin{vmatrix} a, & b, & c, & d \\ a', & b', & c', & d' \\ a'', & b'', & c'', & d'' \\ a''', & b''', & c''', & d''' \end{vmatrix}$$

and then column into column by another constant determinant, the coefficients, all but one of them, of these constant determinants may be used to specialize the form of the resulting equation; which equation will really contain $64-(2\,.\,16-1)=33$ constants; and in order that the quartic surface $(*\,\mathfrak{T}\,\alpha, \beta, \gamma, \delta)^2 = 0$ may have its equation expressible in the form $\Delta=0$, a single relation must hold good among the coefficients: but this in passing.*

67. Returning to the quartic surface

$$\Delta = \begin{vmatrix} A, & B, & C, & D \\ A', & B', & C', & D' \\ A'', & B'', & C'', & D'' \\ A''', & B''', & C''', & D''' \end{vmatrix} = 0,$$

we may connect this not only with the foregoing surface $\Theta=0$, but in a similar manner with another quartic surface $\Phi=0$; viz., taking the current coordinates $(\xi, \eta, \zeta, \omega)$, we may form the lineolinear equations

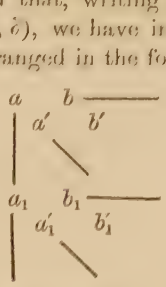
$$\begin{aligned} A\xi + A'\eta + A''\zeta + A'''\omega &= 0, \\ B\xi + B'\eta + B''\zeta + B'''\omega &= 0, \\ C\xi + C'\eta + C''\zeta + C'''\omega &= 0, \\ D\xi + D'\eta + D''\zeta + D'''\omega &= 0, \end{aligned}$$

which, by the elimination of $(\xi, \eta, \zeta, \omega)$, give $\Delta=0$, and by the elimination of $(\alpha, \beta, \gamma, \delta)$ a determinant quartic equation $\Phi=0$ between the coordinates $(\xi, \eta, \zeta, \omega)$; and of course the two surfaces $\Delta=0$, $\Phi=0$ have a point-to-point correspondence such as exists between the surfaces $\Theta=0$, $\Delta=0$. The relation of the point $(\alpha, \beta, \gamma, \delta)$ on the surface $\Delta=0$ to the point (x, y, z, w) on the surface $\Theta=0$, and to the point $(\xi, \eta, \zeta, \omega)$ on the surface $\Phi=0$, may be conveniently indicated by means of the diagram

$$\begin{array}{c} \overbrace{\qquad\qquad\qquad}^{\Theta} \\ x, \quad y, \quad z, \quad w \\ \hline \left.\begin{array}{cccc|c} A, & B, & C, & D & \xi \\ A', & B', & C', & D' & \eta \\ A'', & B'', & C'', & D'' & \zeta \\ A''', & B''', & C''', & D''' & \omega \end{array}\right\} \Phi. \end{array}$$

* Applying the same reasoning to a cubic determinant $\Delta=0$, the number of constants is $36-(2\,.\,9-1)=19$; so that a cubic surface is expressible in the form in question. And so for the quadric determinant $\Delta=0$, the number of constants is $16-(2\,.\,4-1)=9$; so that a quadric surface is expressible in the form in question, as is otherwise obvious.

68. It is to be observed that, writing for A, B, ... their values as linear functions of $(\alpha, \beta, \gamma, \delta)$, we have in all 64 constant coefficients, which we may conceive arranged in the form of a cube, thus:



and taking these in fours height-wise, (a, a_1, a_2, a_3), &c., we compose with them the linear functions $a\alpha + a_1\beta + a_2\gamma + a_3\delta$, &c., which enter into the equation $\Delta = 0$; taking them in fours length-wise, (a, b, c, d), &c., we compose the linear functions $ax + by + cz + dw$, &c., which enter into the equation $\Theta = 0$; and taking them in fours breadth-wise (a, a', a'', a'''), &c., we compose the linear functions $a\xi + a'\eta + a''\zeta + a'''\omega$, &c., which enter into the equation $\Phi = 0$.

69. The process may be indefinitely repeated; we obtain always the same three surfaces over and over again, but on them an indefinite series of corresponding points; viz., we may write

$$\ldots \Theta,\ \Delta,\ \Phi,\quad \Theta, \Delta, \Phi,\quad \Theta,\ \Delta,\ \Phi \ldots$$
$$\ldots \mathrm{P}_1, \mathrm{Q}_1, \mathrm{R}_1,\ \mathrm{P}, \mathrm{Q}, \mathrm{R},\ \mathrm{P}', \mathrm{Q}', \mathrm{R}' \ldots$$

viz., a point Q on Δ corresponds to a point P on Θ and to a point R on Φ; R corresponds to Q on Δ and to a new point P′ on Θ; P′ to R on Φ and to a new point Q′ on Δ, and so on. And in the opposite direction P corresponds to Q on Δ, and to a new point R_1 on Φ; R_1 to P on Θ and to a new point Q_1 on Δ; and so on. And of course the correspondence of any two points of the series, whether belonging to the same surface or to different surfaces, is a one-to-one correspondence.

The Symmetrical Case; Symmetroid and Jacobian.

70. I have established the foregoing general theory; but it is only a particular case of it which connects itself with the theory of nodal quartics; viz., the cube of coefficients is a symmetrically arranged cube

$$\begin{array}{cccc} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{array}$$

$$\begin{array}{cc} a_1 & h_1 \ldots \\ h_1 & b_1 \\ \vdots & \ddots \end{array}$$

or say its upper face is the symmetrical square matrix

$$\left|\begin{array}{cccc} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{array}\right|$$

and the other horizontal planes, the like squares with the several terms affected by suffixes.

The surface $\nabla = 0$ is here a surface of the form

$$\nabla = \left|\begin{array}{cccc} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & P \end{array}\right| = 0$$

[A, B, &c. linear functions of $(\alpha, \beta, \gamma, \delta)$] viz., ∇ is a symmetrical determinant; I call this a symmetroid; the surfaces $=0$, $\Phi = 0$ are one and the same surface, the Jacobian of 4 quadric surfaces; moreover the points P and R are one and the same point, and the correspondence R to P′ is a reciprocal one; so that, instead of the indefinite series of points, we have only 2 points Q, Q′ on the surface ∇, and 2 points P, P′ on the surface Θ $(=\Phi)$; viz., the diagram is

$$\begin{array}{c} \ldots \Delta, \ \Theta, \ \Theta, \ \Delta, \ \Theta, \ \Theta, \ \Delta \ \ldots \\ \ldots Q', \ P', \ P, \ Q, \ P, \ P', \ Q' \ldots \end{array}$$

moreover the symmetroid surface $\nabla = 0$ is a surface with 10 nodes, which is clearly not octadic, and which is therefore the decadianome.

71. Consider the quadric surfaces

$$\begin{array}{lll} S = (a, b, c, d, f, g, h, l, m, n \text{﴿} x, y, z, w)^2 = 0, & & \\ T = (a_1, \ldots & \text{﴿} & \ ,, \ )^2 = 0, \\ U = (a_2, \ldots & \text{﴿} & \ ,, \ )^2 = 0, \\ V = (a_3, \ldots & \text{﴿} & \ ,, \ )^2 = 0, \end{array}$$

and a point $(\alpha, \beta, \gamma, \delta)$ in the same or in a different space, such that the surface $\alpha S + \beta T + \gamma U + \delta V = 0$ is a cone, or say for shortness,

$$\alpha S + \beta T + \gamma U + \delta V = \text{cone};$$

$(\alpha, \beta, \gamma, \delta)$ is said to be the determining point, or determinator of the cone. And if we establish the equations

$$\begin{array}{lll} \delta_x & (\alpha S + \beta T + \gamma U + \delta V) = 0, & \\ \delta_y & (\quad ,, \quad & \ ) = 0, \\ \delta_z & (\quad ,, \quad & \ ) = 0, \\ \delta_w & (\quad ,, \quad & \ ) = 0, \end{array}$$

which express that the surface is a cone, then the point (x, y, z, w) is the vertex of the cone. We have thus 4 equations lincolinear in

(x, y, z, w) and also in $(\alpha, \beta, \gamma, \delta)$, so that the relation between the 2 points is of the nature of that above considered. The relation between (x, y, z, w) is given by the equation

$$\mathrm{J}\,(\mathrm{S}, \mathrm{T}, \mathrm{U}, \mathrm{V}) = 0;$$

viz., the locus is the Jacobian of the 4 quadric surfaces. The relation between $(\alpha, \beta, \gamma, \delta)$ is given by the equation

$$\nabla = \begin{vmatrix} a\alpha + a_1\beta + a_2\gamma + a_3\delta, & h\alpha + \ldots, & g\alpha + \ldots, & l\alpha + \ldots \\ h\alpha + \ldots, & b\alpha + \ldots, & f\alpha + \ldots, & m\alpha + \ldots \\ g\alpha + \ldots, & f\alpha + \ldots, & e\alpha + \ldots, & n\alpha + \ldots \\ l\alpha + \ldots, & m\alpha + \ldots, & n\alpha + \ldots, & d\alpha + \ldots \end{vmatrix} = 0,$$

so that the locus is (by the foregoing definition) the symmetroid. And the determinator point on the symmetroid thus corresponds to the cone-vertex on the Jacobian.

72. But the Jacobian may be obtained in a different manner; viz., if we establish the equations

$$\begin{aligned} (\xi\delta_x + \eta\delta_y + \zeta\delta_z + \omega\delta_w)\,\mathrm{S} &= 0, \\ (\quad\quad ,,\quad\quad)\,\mathrm{T} &= 0, \\ (\quad\quad ,,\quad\quad)\,\mathrm{U} &= 0, \\ (\quad\quad ,,\quad\quad)\,\mathrm{V} &= 0, \end{aligned}$$

then the elimination of $(\xi, \eta, \zeta, \omega)$ leads to the equation $\mathrm{J}\,(\mathrm{S}, \mathrm{T}, \mathrm{U}, \mathrm{V}) = 0$ of the Jacobian surface. And since each of the equations is symmetrical in regard to (x, y, z, w) and $(\xi, \eta, \zeta, \omega)$, it appears that the point $(\xi, \eta, \zeta, \omega)$ is also a point on the Jacobian surface. We have on the symmetroid a point related to $(\xi, \eta, \zeta, \omega)$ in the same way that $(\alpha, \beta, \gamma, \delta)$ on the symmetroid is related to the point (x, y, z, w); and this completes the system of the 4 points, Q on the symmetroid, P and P′ on the Jacobian, Q′ on the symmetroid; but in what follows I make no use of this last point Q′.

73. The points (x, y, z, w), $(\xi, \eta, \zeta, \omega)$ on the Jacobian correspond in such wise that, taking the polar planes of either of them in regard to the quadrics $\mathrm{S} = 0$, $\mathrm{T} = 0$, $\mathrm{U} = 0$, $\mathrm{V} = 0$, these intersect in a single point, viz., in the other of the two corresponding points. Or, what is the same thing, the line joining the two points cuts each of the four quadrics harmonically, whence also it cuts harmonically any quadric surface whatever of the series $\alpha\mathrm{S} + \beta\mathrm{T} + \gamma\mathrm{U} + \delta\mathrm{V} = 0$, ($\alpha$, β, γ, δ being here arbitrary multipliers); viz., this property is an immediate interpretation of the equation

$$(\xi\delta_x + \eta\delta_y + \zeta\delta_z + \omega\delta_w)\,(\alpha\mathrm{S} + \beta\mathrm{T} + \gamma\mathrm{U} + \delta\mathrm{V}) = 0,$$

or, as this is more conveniently written,

$$(\alpha, \ldots \mathrm{X}\xi, \eta, \zeta, \omega \mathrm{X} x, y, z, w) = 0,$$

if for a moment $(a, \ldots)$ denote the coefficients of the quadric function $\alpha S+\beta T+\gamma U+\delta V$.

74. Consider any 6 pairs of points (x_1, y_1, z_1, w_1), $(\xi_1, \eta_1, \zeta_1, \omega_1)$, &c., related as above; the quadric surfaces $S = 0$, $T = 0$, $U = 0$, $V = 0$ are surfaces cutting harmonically the lines joining the two pairs of points respectively; or say they are quadrics cutting harmonically 6 given segments; and the general quadric surface which cuts harmonically the 6 given segments is $\alpha S+\beta T+\gamma U+\delta V = 0$. We thus see that the Jacobian surface $J(S, T, U, V) = 0$ is in fact the locus of the vertices of the quadric cones which cut harmonically 6 given segments. The surface so defined by M. Chasles (Comptes Rendus, tom. lii., 1861, pp. 1157–62), and shown by him to be a quartic surface, is thus identified with the Jacobian of any 4 quartic surfaces; and included herein we have the particular case, also considered by him, of the locus of the vertices of the quadric cones which pass through 6 given points, or Jacobian of the 6 given points.

75. It is to be shown that there are 10 systems of values $(\alpha, \beta, \gamma, \delta)$, or, what is the same thing, 10 points on the symmetroid, for each of which the quadric surface $\alpha S+\beta T+\gamma U+\delta V = 0$ is a plane-pair. For any such system of values the plane-pair may be regarded as a cone, having its vertex at any point whatever on the line which is the axis of the plane-pair; that is, each point of this line is the vertex of a cone of the system of surfaces $\alpha S+\beta T+\gamma U+\delta V = 0$; or, what is the same thing, the axis of the plane-pair lies on the Jacobian surface; viz., there will be on the Jacobian surface 10 lines. Moreover, to the point $(\alpha, \beta, \gamma, \delta)$ on the symmetroid there corresponds indifferently any point whatever on the axis of the plane-pair. The analytical expressions for (x, y, z, w) in terms of $(\alpha, \beta, \gamma, \delta)$ must therefore, for the values in question of $(\alpha, \beta, \gamma, \delta)$, become indeterminate; and this can only happen if for the values in question the first minors of the determinant ∇ all of them vanish. But a point $(\alpha, \beta, \gamma, \delta)$, for which the minors of ∇ all of them vanish, is obviously a node on the symmetroid; and it thus appears that there are on the symmetroid 10 nodes, each corresponding to a line on the Jacobian, and that the condition for determining these is

$$\alpha S+\beta T+\gamma U+\delta V = \text{plane-pair};$$

viz., the values of $(\alpha, \beta, \gamma, \delta)$, which satisfy this condition, belong to a node of the symmetroid, and the line on the Jacobian is the axis of the plane-pair.

76. Reverting to the equation $\nabla = 0$ of the symmetroid, where ∇ is a symmetrical determinant the terms of which are linear functions of the coordinates $(\alpha, \beta, \gamma, \delta)$, it has already been shown, *ante* No. 7, that

this is a surface with 10 nodes; but this may be also proved as follows. Writing as before

$$\alpha S+\beta T+\gamma U+\delta V = (A, B, C, D, F, G, H, L, M, N\chi x, y, z, w)^2 = 0,$$

the condition that this shall be a plane-pair implies a three-fold relation between the coefficients A, B, &c., and the required number of nodes is equal to the order of this three-fold relation. Establishing between the coefficients A, B, &c., any 6 linear relations whatever, we should have a nine-fold relation to determine the ratios of the 10 quantities; and the number of solutions would be equal to the order of the three-fold relations. But taking the 6 linear relations to be of the form $(A, \ldots\chi x_1, y_1, z_1, w_1)^2 = 0$, the question is in fact to find the number of the plane-pairs which pass through 6 given points; and this is clearly $= 10$.

77. Applying the conclusion to the system of quadric surfaces $\alpha S+\beta T+\gamma U+\delta V = 0$, we see that there are in the system 10 plane-pairs; and that the lines of intersection, or axes of the plane-pairs, are lines upon the Jacobian surface.

78. The equation $\nabla = 0$ of the symmetroid seems to contain homogeneously 40 constants. But starting with any given symmetrical determinant, we may multiply it line into line by a constant determinant, and then column into column by the same constant determinant, in such wise that the resulting product is still a symmetrical determinant; and the coefficients of the constant determinant may then be used to specialise the form of the equation. The equation $\nabla=0$ of the symmetroid thus really contains $40-16 = 24$ constants; this is as it should be, for the symmetroid, *quà* quartic surface with 10 nodes, contains $34-10 = 24$ constants.

Symmetroid with given Nodes.

79. A symmetroid can be formed with 7 given points as nodes; but there is no proper symmetroid with 8 given points as nodes. If we endeavour to form such a symmetroid, we obtain a system of 2 quadric cones, each of them passing through the 8 points; viz., these are any 2 out of the 4 quadric cones which pass through the 8 points. This will be shown in a moment; for the complete *à posteriori* identification with the decadianome, it would be necessary to show that a symmetroid could be found having for nodes 7 given points, an 8th point anywhere on the dianodal surface, and a 9th point anywhere on the dianodal curve; but this I have not succeeded in effecting.

80. We have for any node $(\alpha, \beta, \gamma, \delta)$ of the symmetroid,

$$\alpha S+\beta T+\gamma U+\delta V = \text{plane-pair}.$$

If, then, 4 of the given nodes are (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), we must have S, T, U, V each of them a plane-pair. We may without loss of generality assume $S = x^2+y^2$, $T = z^2+w^2$; this, however, does not determine the signification of the coordinates (x, y, z, w), for S will remain unaltered if we write therein

$$x\cos\theta+y\sin\theta,\ x\sin\theta-y\cos\theta \text{ for } x, y;$$

and similarly T will remain unaltered if we write therein

$$z\cos\theta_1+w\sin\theta_1,\ z\sin\theta_1-w\cos\theta_1 \text{ for } z, w.$$

Hence, if we go on to assume

$$U = k\,(x+my+nz+pw)\ (x+m'y+n'z+p'w),$$
$$V = k_1(x+m_1y+n_1z+p_1w)\,(x+m_1'y+n_1'z+p_1'w),$$

we may imagine the θ, θ_1 so determined that, for instance,

$$m+m' = 0,\quad p_1+p'_1 = 0;$$

we have thus

$$S = x^2+y^2$$
$$T = \quad z^2+w^2$$
$$U = k\,(x+my+nz+pw)\,(x-my+n'z+p'w)$$
$$V = k_1(x+m_1y+n_1z+p_1w)\,(x+m_1'y+n_1'z-p_1w);$$

formulæ which contain the 12 constants

$$(k, m, n, p, n', p', k_1, m_1, n_1, p_1, m_1', n_1').$$

This is right, for the symmetroid containing 24 constants, the symmetroid with 4 given nodes should contain $(24-4\,.\,3 =)$ 12 constants. And each additional given node will determine 3 constants: hence for 4 new given nodes the expressions become determinate (not of necessity uniquely so).

81. But for any 4 new nodes, the equations may be satisfied by writing therein $n=n'$, $p=-p'$, $m=-m_1'$, $n_1=n_1'$; viz., they then assume the form

$$S = x^2+\quad y^2$$
$$T = \quad z^2\quad +\ w^2$$
$$U = (ax+cz)^2+(by+dw)^2$$
$$V = (a'x+c'z)^2+(b'y+d'w)^2,$$

containing 8 constants, which may be determined so that the nodes shall be the 4 given points. If now with the last mentioned values we form the value of $\alpha S+\beta T+\gamma U+\delta V$, this will consist of two terms $(*\,\mathfrak{J}x, z)^2$ and $(*\,\mathfrak{J}y, w)^2$, the first of which will be a square if

$(\alpha+\gamma a^2+\delta a'^2)\,(\beta+\gamma c^2+\delta c'^2)-(\gamma ac+\delta a'c')^2 = 0$, say this is $\Lambda = 0$,

and the second will be a square if

$(\alpha+\gamma b^2+\delta b'^2)\,(\beta+\gamma d^2+\delta d'^2)-(\gamma bd+\delta b'd')^2 = 0$, say this is $\Lambda' = 0$;

so that the condition

$$\alpha S+\beta T+\gamma U+\delta V = \text{cone}$$

will be satisfied if $\Lambda=0$, or if $\Lambda'=0$; that is, the equation of the sym-

metroid will be $\Lambda\Lambda'=0$, or the symmetroid breaks up into the 2 quadric surfaces $\Lambda=0$, $\Lambda'=0$, each of which is a cone.

82. It is to be further observed that, considering the first mentioned 4 points $(1,0,0,0)$, &c., and any other 4 given points whatever, the equation of any one of the 4 quadric cones through these 8 points will be of the form $(*\,\alpha\beta\gamma, \gamma\alpha, \alpha\beta, \alpha\delta, \beta\delta, \gamma\delta) = 0$;

viz., any equation of this form, being a cone, will admit of being expressed, and that in one way only, in the form $\Lambda=0$. Consider then any one of the 4 cones through the 8 points, and let its equation be thus expressed; we have the values of the coefficients a, c, a', c', which enter into the expressions of S, T, U, V; and similarly, considering any other of the 4 cones, and expressing its equation in the like form, we have the values of the coefficients b, d, b', d' which enter into the expressions of S, T, U, V.

83. If instead of taking 2 different cones through the 8 points, we take in each case the same cone, the expressions for S, T, U, V would be

$$\begin{aligned} \mathrm{S} &= x^2 + y^2 \\ \mathrm{T} &= z^2 + w^2 \\ \mathrm{U} &= (ax+cz)^2 + (ay+cw)^2 \\ \mathrm{V} &= (a'x+c'z)^2 + (a'y+c'w)^2; \end{aligned}$$

and we have identically

$$(ac'-a'c)(aa'\mathrm{S}-cc'\mathrm{T}) - a'c'\mathrm{U} + ac\mathrm{V} = 0.$$

This solution may be disregarded.

84. Instead of the assumption $\mathrm{S}=x^2+y^2$, $\mathrm{T}=z^2+w^2$, we may take $x=0$, $y=0$, $z=0$, $w=0$ to be planes of the plane-pairs S, T, U, V respectively; it is then easy to fix the remaining constants so that the 5th and 6th nodes of the symmetroid shall be given points. Suppose that the coordinates of the 5th node are $(1,1,1,1)$; to obtain the result in the most simple manner, I take for the moment Ω an arbitrary quadric function $(x,y,z,w)^2$, and I write

$$\begin{aligned} \mathrm{S} &= x(\delta_x\Omega \quad + hy - gz + aw), \\ \mathrm{T} &= y(\delta_y\Omega - hx \quad + fz + bw), \\ \mathrm{U} &= z(\delta_z\Omega + gx - fy \quad + cw), \\ \mathrm{V} &= w(\delta_w\Omega - ax - by - cz \quad), \end{aligned}$$

where the coefficients are arbitrary. We have identically $\mathrm{S}+\mathrm{T}+\mathrm{U}+\mathrm{V} = 2\Omega$; wherefore the given point $(1,1,1,1)$ will be a node of the symmetroid, if only $\Omega=0$ be a plane-pair; and it is easy to see that we may without loss of generality take one factor to be $x+y+z+w$, and write

$$\Omega = (x+y+z+w)(lx+my+nz+pw);$$

viz., Ω having this value, the symmetroid, $a\mathrm{S}+\beta\mathrm{T}+\gamma\mathrm{U}+\delta\mathrm{V} =$ cone, will have the 5 given nodes; the equation contains, as it should do, 9 constants.

85. In order that the symmetroid may have a 6th given node $(\alpha_1, \beta_1, \gamma_1, \delta_1)$, I observe that the constants can be determined so that $\alpha_1 S + \beta_1 T + \gamma_1 U + \delta_1 V$ shall be equal to an arbitrary quadric function, say

$$\alpha_1 S + \beta_1 T + \gamma_1 U + \delta_1 V = (a, b, c, d, f, g, h, l, m, n \mathbin{\text{\textbigtriangledown}} x, y, z, w)^2;$$

this in fact gives $\quad (l, m, n, p) = \left(\frac{a}{\alpha_1}, \frac{b}{\beta_1}, \frac{c}{\gamma_1}, \frac{d}{\delta_1}\right);$

and then, completing the comparison,

$$S = x\left\{\left[\frac{a}{\alpha_1}\right]x + \left[\frac{2h}{\alpha_1-\beta_1} - \frac{\beta_1}{\alpha_1-\beta_1}\left(\frac{a}{\alpha_1}+\frac{b}{\gamma_1}\right)\right]y + \left[\frac{2g}{\alpha_1-\gamma_1} - \frac{\gamma_1}{\alpha_1-\gamma_1}\left(\frac{a}{\alpha_1}+\frac{c}{\gamma_1}\right)\right]z + \left[\frac{2l}{\alpha_1-\delta_1} - \frac{\delta_1}{\alpha_1-\delta_1}\left(\frac{a}{\alpha_1}+\frac{d}{\delta_1}\right)\right]w\right\},$$

$$T = y\left\{\left[\frac{2h}{\beta_1-\alpha_1} - \frac{\alpha_1}{\beta_1-\alpha_1}\left(\frac{b}{\beta_1}+\frac{a}{\alpha_1}\right)\right]x + \left[\frac{b}{\beta_1}\right]y + \left[\frac{2f}{\beta_1-\gamma_1} - \frac{\gamma_1}{\beta_1-\gamma_1}\left(\frac{b}{\beta_1}+\frac{c}{\gamma_1}\right)\right]z + \left[\frac{2m}{\beta_1-\delta_1} - \frac{\delta_1}{\beta_1-\delta_1}\left(\frac{b}{\beta_1}+\frac{d}{\delta_1}\right)\right]w\right\},$$

$$U = z\left\{\left[\frac{2g}{\gamma_1-\alpha_1} - \frac{\alpha_1}{\gamma_1-\alpha_1}\left(\frac{c}{\gamma_1}+\frac{a}{\alpha_1}\right)\right]x + \left[\frac{2f}{\gamma_1-\beta_1} - \frac{\beta_1}{\gamma_1-\beta_1}\left(\frac{c}{\gamma_1}+\frac{b}{\beta_1}\right)\right]y + \left[\frac{c}{\gamma_1}\right]z + \left[\frac{2n}{\gamma_1-\delta_1} - \frac{\delta_1}{\gamma_1-\delta_1}\left(\frac{c}{\gamma_1}+\frac{d}{\delta_1}\right)\right]w\right\},$$

$$V = w\left\{\left[\frac{2l}{\delta_1-\alpha_1} - \frac{\alpha_1}{\delta_1-\alpha_1}\left(\frac{d}{\delta_1}+\frac{a}{\alpha_1}\right)\right]x + \left[\frac{2m}{\delta_1-\beta_1} - \frac{\beta_1}{\delta_1-\beta_1}\left(\frac{d}{\delta_1}+\frac{b}{\beta_1}\right)\right]y + \left[\frac{2n}{\delta_1-\gamma_1} - \frac{\gamma_1}{\delta_1-\gamma_1}\left(\frac{d}{\delta_1}+\frac{c}{\gamma_1}\right)\right]z + \left[\frac{d}{\delta_1}\right]w\right\};$$

viz., these values give

$$S + T + U + V = (x+y+z+w)\left(\frac{a}{\alpha_1}x + \frac{b}{\beta_1}y + \frac{c}{\gamma_1}z + \frac{d}{\delta_1}w\right),$$

$$\alpha_1 S + \beta_1 T + \gamma_1 U + \delta_1 V = (a, b, c, d, f, g, h, l, m, n \mathbin{\text{\textbigtriangledown}} x, y, z, w);$$

hence, taking the function $(a, \ldots. \mathbin{\text{\textbigtriangledown}} x, y, z, w)^2$ to be a plane-pair equal to $(x + iy + jz + kw)(x + i_1 y + j_1 z + k_1 w)$ suppose, or considering the coefficients $(a, \ldots.)$ as given functions of (i, j, k, i_1, j_1, k_1), we have the symmetroid having the 6 given nodes and containing the last mentioned 6 constants.

The Jacobian with given Lines.

86. The Jacobian contains 24 constants; obviously it is uniquely determined if 4 of the plane-pairs thereof are given; and it is also determined, but not uniquely, if 6 of the lines thereof are given. We may inquire how many given nodes of the symmetroid may be considered as corresponding to given plane-pairs, or lines of the Jacobian. Take as given any 4 nodes of the symmetroid; the corresponding 4 plane-pairs may be taken to be given plane-pairs; and we may besides take as given a 5th node of the symmetroid. For let the first 4 nodes of the symmetroid be $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$; the given plane-pairs $P_1Q_1 = 0$, $P_2Q_2 = 0$, $P_3Q_3 = 0$, $P_4Q_4 = 0$; (l_1, l_2, l_3, l_4) any system of values such that we have

$$l_1P_1Q_1 + l_2P_2Q_2 + l_3P_3Q_3 + l_4P_4Q_4 = \text{plane-pair};$$

and $(1, 1, 1, 1)$ the 5th node of the symmetroid; we have only to assume

$$(S, T, U, V) = (l_1P_1Q_1,\ l_2P_2Q_2,\ l_3P_3Q_3,\ l_4P_4Q_4).$$

87. Suppose, however, that on the Jacobian we have given, not the 4 plane-pairs, but only the 4 axes of the plane-pairs; the plane-pairs may be taken to be

$$(1, b_1, c_1\nabla P_1, Q_1)^2 = 0, \ldots\ldots (1, b_4, c_4\nabla P_4, Q_4)^2 = 0,$$

where the 8 constants $(b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4)$ are in the first instance undetermined. If we attempt to find l_1, l_2, l_3, l_4, so that

$$l_1\,(1, b_1, c_1\nabla P_1, Q_1)^2 \ldots + l_4\,(1, b_4, c_4\nabla P_4, Q_4)^2 = \text{plane-pair of given axis},$$

we have between the coefficients (b, c) 4 equations; and similarly, if we attempt to find m_1, m_2, m_3, m_4 such that

$$m_1\,(1, b_1, c_1\nabla P_1, Q_1)^2 \ldots + m_4\,(1, b_4, c_4\nabla P_4, Q_4)^2 = \text{plane pair of another given axis},$$

we have 4 more equations between the coefficients (b, c); viz., these will be determined by the 8 equations (this is in fact the before mentioned property that 6 lines of the Jacobian may be taken to be given lines). But considering only the first system of equations; in order that to the given axis may correspond a given node on the symmetroid, say the node $(1, 1, 1, 1)$, we have only to write

$$S = l_1\,(1, b_1, c_1\nabla P_1, Q_1)^2 : \ldots\ldots V = l_4\,(1, b_4, c_4\nabla P_4, Q_4)^2;$$

that is, we may take as given 5 nodes of the symmetroid, and the corresponding 5 lines of the Jacobian; the formulæ will contain 4 constants; we may by means of them make the Jacobian have a 6th given line, thus determining the constants; or we may make the symmetroid have a 6th given node, leaving in this case one constant arbitrary.

Correspondence on the Jacobian : Lines and Skew Cubics.

88. I consider the correspondence of two points on the Jacobian ; it is to be shown that when one of the points is on a line of the Jacobian, the corresponding point will be on a skew cubic ; that is, that corresponding to each line of the Jacobian we have (on the Jacobian) a skew cubic. Call the plane-pairs of the system of quadric surfaces $1, 2, 3 \ldots 10$; selecting any 4 of these, say $1, 2, 3, 4$, the polar planes of any point of the Jacobian in regard to these 4 plane-pairs will meet in a point which will be the required corresponding point. And observe that, in regard to any one of the plane-pairs, say 1, the polar plane of a point P is the plane through the axis harmonic to the plane through the axis and the point P. Hence, for a point on the axis of 1, the polar plane in regard to 1 is indeterminate ; the polar planes in regard to the plane-pairs $2, 3, 4$ respectively meet in a point which is the required corresponding point. We may for any point whatever take the polar planes in regard to the plane-pairs $2, 3, 4$ respectively, and call the intersection of these planes the corresponding point ; this being so, if the first mentioned point moves along a line, the corresponding point moves along a curve, which is easily shown to be a skew cubic cutting the axis of each plane-pair twice ; that is, in regard to the plane-pairs $2, 3, 4$, the locus corresponding to any line whatever is a skew cubic cutting the axis of each plane-pair twice. In particular, the corresponding curve of the axis of 1, is a skew cubic cutting the axis of the plane-pairs $2, 3, 4$ each twice ; but the axis of 1 does not stand in any special relation to the plane-pairs $2, 3, 4$, as distinguished from the remaining plane-pairs $5, 6 \ldots 10$: we have therefore the more complete theorem, that the skew cubic cuts the axes of the plane-pairs $2, 3 \ldots 10$ each twice ; or, instead of the plane-pairs, speaking of the line $1, 2, 3 \ldots 10$, we may say that corresponding to any one of the lines we have a skew cubic meeting the other 9 lines each of them twice.

89. I stop for a moment to prove the subsidiary theorem assumed in the foregoing demonstration. Let the 3 plane-pairs be $PQ=0$, $RS=0$, $TU=0$, and let the line be that joining the points (x_0, y_0, z_0, w_0) and (x_1, y_1, z_1, w_1) ; the coordinates of any point in the line may be taken to be $\lambda x_0+\mu x_1,\ \lambda y_0+\mu y_1,\ \lambda z_0+\mu z_1,\ \lambda w_0+\mu w_1$; and hence for the polar plane in regard to the plane-pair $PQ=0$ we have

$$\{(\lambda x_0+\mu x_1)\delta_x \ldots + (\lambda w_0+\mu w_1)\delta_w\}\ PQ = 0 ;$$

viz., this equation may be written

$$\lambda\,(PQ_0+P_0Q)+\mu\,(PQ_1+P_1Q) = 0 ;$$

forming the like equations in regard to the other 2 plane-pairs respectively, and eliminating λ, μ, we obtain for the required locus

$$\left\| \begin{matrix} PQ_0+P_0Q, & RS_0+R_0S, & TU_0+T_0U \\ PQ_1+P_1Q, & RS_1+R_1S, & TU_1+T_1U \end{matrix} \right\| = 0,$$

a skew cubic; and on writing herein $P=0$, $Q=0$, the equations become

$$\left| \begin{matrix} RS_0+R_0S, & TU_0+T_0U \\ RS_1+R_1S, & TU_1+T_1U \end{matrix} \right| = 0;$$

viz., the line ($P=0$, $Q=0$) meets the skew cubic in the points where the line meets the quadric surface determined by this last equation, that is, in 2 points.

90. We have thus on the Jacobian the 10 lines $1, 2, \ldots 9, 10$, and corresponding thereto respectively the 10 skew cubics $1', 2', \ldots 9', 10'$, where each line meets twice each of the skew cubics except that denoted by the same number; a relation similar to that which exists between the lines $1, 2, 3, 4, 5, 6$ and $1', 2', 3', 4', 5', 6'$, which compose a double-sixer on a cubic surface.

Suppose that there are given on the Jacobian the lines $1, 2, 3, 4, 5, 6$; meeting each of these twice, we have the skew cubics $7', 8', 9', 10'$; and then

	7		$8', 9', 10'$
the lines	8	meet twice each of the cubics	$9', 10', 7'$
	9		$10', 7', 8'$
	10		$7', 8', 9'$

so that the determination of the remaining 4 lines depends upon that of the skew cubics $7', 8', 9', 10'$, which meet each of the given lines twice.

91. To determine a skew cubic cutting twice each of 6 given lines, I proceed as follows. Let the lines be $1, 2, 3, 4, 5, 6$; take $U=0$ the general quadric surface through the lines 1 and 2, $V=0$ the general quadric surface through the lines $1, 3$ (the equations contain each of them homogeneously 4 constants). The 2 surfaces intersect in the line 1, and in a skew cubic cutting twice each of the lines $1, 2, 3$; we have therefore to determine the constants so that the 2 surfaces may meet the line 4 in the same 2 points, the line 5 in the same 2 points, the line 6 in the same 2 points. Imagine for a moment the equations of any one of the lines $4, 5, 6$ to be $z=0$, $w=0$; the equations of the 2 surfaces, substituting therein these values, would assume the forms

$$(a, b, c\,\text{X}\,x, y)^2 = 0, \quad (a', b', c'\,\text{X}\,x, y)^2 = 0;$$

and the conditions for the intersection in the same 2 points would be $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, $=p$ suppose. This is in fact the form of the conditions, understanding a, b, c to be linear functions of the coefficients of U, and a', b', c' to be linear functions of the coefficients of V. We have in this manner 3 sets of equations involving respectively the indeterminate quantities p, q, r; viz., these may be represented by

$a=pa'$, $b=pb'$, $c=pc'$; $d=qd'$, $e=qe'$, $f=qf'$; $g=rg'$, $h=rh'$, $i=ri'$; where the unaccented letters $a, b \ldots i$ are linear functions of the co-

efficients of U, and the accented letters $a', b' \ldots i'$ linear functions of the coefficients of V. Eliminating the coefficients of U, V, we have between p, q, r a twofold relation, which may be represented as follows:

$$\begin{Vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ p & p & p & q & q & q & r & r & r \\ p & p & p & q & q & q & r & r & r \\ p & p & p & q & q & q & r & r & r \\ p & p & p & q & q & q & r & r & r \end{Vmatrix} = 0,$$

it being understood that the 1's represent constants, and the p's, q's, and r's linear functions of these variables respectively. The several equations of the system, regarding therein p, q, r as coordinates, represent each of them a quartic curve; any 2 of these intersect in 16 points; but the number of points common to all the curves is = 10. But each of the curves passes through the 3 points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; these are consequently included among the 10 points, but they do not give a proper solution of the question; and the number of solutions is thus reduced to $10-3 = 7$. There is yet another solution to be rejected; viz., U=0 being a quadric surface through the lines 1, 2, and V=0 the quadric surface through the lines 1, 3, it is possible to determine the coefficients of U, V so that each of these surfaces shall be the quadric surface through the lines 1, 2, 3; and if we then have identically $U = \theta V$, it is clear that corresponding values of p, q, r are $p=q=r\ (=\theta)$. We have thus the point $p=q=r$ common to all the curves of the system; this solution counts, I believe, once only, and the number of relevant solutions is $7-1 = 6$.

92. It may be observed, in regard to the foregoing solution, that if we take $123 = 0$ as the equation of the quadric surface through the lines 1, 2, 3, and so in other cases, then the equation of the surfaces U=0 and V=0 may be taken to be

$$\lambda . 123 + \mu . 124 + \nu . 125 + \rho . 126 = 0,$$
$$\lambda' . 132 + \mu' . 134 + \nu' . 135 + \rho' . 136 = 0,$$

respectively, the coefficients of the two surfaces being here put in evidence. And it is clear that for $\mu=\nu=\rho=0$, $\mu'=\nu'=\rho'=0$, the surfaces become each of them the surface through the lines 1, 2, 3.

93. The conclusion is, that touching twice each of the six lines 1, 2, 3, 4, 5, 6, we have six skew cubics; it would appear that any four of these may be taken for the skew cubics $7', 8', 9', 10'$ (so that there are 15 such tetrads of cubics). I am not, however, able to verify that we then have the remaining 4 lines each cutting twice 3 of the 4 skew cubics; assuming that for each system of 4 skew cubics there

is one, and only one, such system of lines, then of course to the given system of lines 1, 2, 3, 4, 5, 6, there will belong 15 systems of lines 7, 8, 9, 10, and therefore also 15 Jacobian surfaces.

Further Investigations as to the Jacobian, &c.

94. Taking $(\xi, \eta, \zeta, \omega)$ as plane-coordinates, two quadric surfaces

$$(a, b, c, d, f, g, h, l, m, n\,\mathcal{J}\,\xi, \eta, \zeta, \omega)^2 = 0$$

and

$$(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{F}, \mathrm{G}, \mathrm{H}, \mathrm{L}, \mathrm{M}, \mathrm{N}\,\mathcal{J}\,x, y, z, w)^2 = 0$$

are said to be interverts (or interverse) one of the other, when we have between the coefficients the relation

$$(a, b, c, d, f, g, h, l, m, n\,\mathcal{J}\,\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{F}, \mathrm{G}, \mathrm{H}, \mathrm{L}, \mathrm{M}, \mathrm{N}) = 0,$$

that is,

$$a\mathrm{A} + \ldots + 2f\mathrm{F} + \ldots = 0.$$

The condition that the two surfaces may be interverts of each other is linear in regard to the coefficients of each surface separately; hence, using a before explained locution, we may say—interverse to a given quadric surface we have 9 quadrics; interverse to two given quadrics 8 quadrics; or generally, that interverse to k given quadrics we have $10-k$ quadrics. And, moreover, if the quadrics of the two systems be $\mathrm{L}=0$, $\mathrm{M}=0$, &c., and $\mathrm{S}=0$, $\mathrm{T}=0$, $\mathrm{U}=0$, &c., then every quadric $\lambda\mathrm{L}+\mu\mathrm{M}+\ldots=0$ is interverse to each of the quadrics $\alpha\mathrm{S}+\beta\mathrm{T}+\gamma\mathrm{U}+\ldots=0$.

If the quadric $(a, \ldots\,\mathcal{J}\,\xi, \eta, \zeta, \omega)^2 = 0$ be an intervert of the plane-pair $(lx+my+nz+p\omega\,\mathcal{J}\,l'x+m'y+n'z+p'w) = 0$, the condition is

$$(a, \ldots\,\mathcal{J}\,l, m, n, p\,\mathcal{J}\,l', m', n', p') = 0;$$

viz., this expresses that the two planes are harmonics in regard to the pair of planes drawn through the axis of the plane-pair to touch the quadric surface; or say, that the plane-pair is harmonic in regard to the quadric.

95. To apply this to the Jacobian surface, I recall that, starting with the given quadric surfaces $\mathrm{S}=0$, $\mathrm{T}=0$, $\mathrm{U}=0$, $\mathrm{V}=0$, and taking $(\alpha, \beta, \gamma, \delta)$ to be such that

$$\alpha\mathrm{S}+\beta\mathrm{T}+\gamma\mathrm{U}+\delta\mathrm{V} = \text{plane-pair},$$

there are 10 such plane-pairs, and that the axes of these are the lines of the Jacobian. If instead of the given quadric surfaces, we consider the 6 interverse surfaces $(a_1, \ldots\,\mathcal{J}\,\xi, \eta, \zeta, \omega)^2 = 0$, $\ldots (a_6, \ldots\,\mathcal{J}\,\xi, \eta, \zeta, \omega)^2 = 0$, then the condition is that the plane-pair shall be harmonic in regard to each of these surfaces. Let the quadric surfaces be called 1, 2, 3, 4, 5, 6; then, attending to any three of these, say 1, 2, 3, the plane-pair is harmonic in regard to these three surfaces. Through the axis of the plane-pair draw tangent planes to 1, 2, and 3 respectively; each of these pairs of planes is harmonic in regard to the planes of the plane-pair; that is, the three pairs of tangent planes are in involution; or, as

we may also express it, the axis is (*quoad* its planes) in involution in regard to the three quadric surfaces. Conversely, when the axis is thus in involution in regard to the surfaces 1, 2, and 3, we may by means of the surfaces 1 and 2 determine the two planes of the plane-pair, and then these will be harmonics in regard to the surface 3. It thus appears that the axis is given as a line which is (*quoad* its planes) in involution in regard to the surfaces 1, 2, 3, to the surfaces 1, 2, 4, the surfaces 1, 2, 5, and the surfaces 1, 2, 6, respectively; or, as we may express it, as a line which is (*quoad* its planes) in involution in regard to the surfaces 1, 2, 3, 4, 5, 6.

96. It is substantially the same thing, but it is rather easier, to consider the whole question under the reciprocal form; viz., instead of a plane-pair and a quadric surface represented by an equation in plane-coordinates, to take a point-pair and a quadric surface represented by an equation in point-coordinates; we have thus a line which is (*quoad* its points) in involution in regard to three given quadric surfaces, or as we may more simply express it, which cuts in involution the three given surfaces; and we thus arrive at the problem of finding a line which cuts in involution six given quadric surfaces; viz., this is equivalent to the above problem where the line has to satisfy (*quoad* its planes) the like condition; and in each problem the number of solutions should be $= 10$.

97. Consider a line which cuts in involution the three given surfaces $(a_1, \ldots \Sigma x, y, z, w) = 0, (a_2, \ldots \Sigma x, y, z, w)^2 = 0, (a_3, \ldots \Sigma x, y, z, w)^2 = 0$. I will presently show that this implies a cubic relation $(* \Sigma a, b, c, f, g, h)^3$ between the six coordinates of the line. But assuming it for the moment, suppose that the line cuts in involution the three surfaces and a fourth quadric surface $(a_4, \ldots \Sigma x, y, z, w)^2 = 0$. Considering the line as cutting in involution the surfaces 1, 2, 4, we have between the six coordinates a second cubic relation; there is, however, a reduction, and the order of the resulting two-fold relation between the coordinates is $3 \cdot 3 - 4 = 5$. To explain this, observe that every line which cuts in the same two points the surfaces 1 and 2 respectively (that is, which cuts the curve of intersection twice) will in an improper sense cut in involution the surfaces 1, 2, 3, and also the surfaces 1, 2, 4. There is thus a reduction equal to the order in the six coordinates of the two-fold relation which expresses that the line cuts twice the curve of intersection of the surfaces 1 and 2. Join hereto the relations that the line meets each of two given lines; the coordinates of the line are determined by the twofold relation (say its order is $= \lambda$) two linear equations, and the universal equation $af + bg + ch = 0$; the number of solutions is $= 2\lambda$. But the number of solutions is equal to that of the lines which meet the quadriquadric curve of intersection twice, and

meet also each of two given lines; or what is the same thing, it is equal to the order of the scroll generated by the lines which meet the curve twice, and also a given line. We have for the curve of intersection (m the order, h the number of apparent double points) $m = 4$, $h = 2$; whence order of the scroll is $2 + \frac{1}{2}4 \cdot 3 = 8$; that is, $2\lambda = 8$, or $\lambda = 4$, which is the required reduction.

98. If the line cut in involution 5 given quadric surfaces [say the 5th surface is $(a_5, \ldots \Sigma x, y, z, w)^2 = 0$]; then we have between the 6 coordinates a three-fold relation, the order of which is $3 \cdot 5 -$reduction. This should be $= 10$, and consequently the reduction $= 5$; for admitting the value to be 10, the order (in the ordinary sense) of the scroll generated by the lines which cut in involution the 5 given quadrics should be $= 20$; and conversely. But the value 20 may be verified without difficulty. For the question may be transformed as follows:— If a point-pair be harmonic in regard to each of 5 given quadrics, how many of the axes (or lines through the 2 points of a point-pair) cut a given line. Take (x, y, z, w), (x', y', z', w') as the coordinates of the 2 points of a point-pair; the harmonic condition in regard to a quadric surface $U = 0$ is $x'\delta_x U + y'\delta_y U + z'\delta_z U + w'\delta_w U = 0$ [where U is regarded as a function of the (x, y, z, w) belonging to a point of the point-pair]; the condition for the intersection with a given line is a lineolinear equation in the coordinates (x, y, z, w) and (x', y', z', w'), or say it is $Lx' + My' + Nz' + Pw' = 0$, where L, M, N, P are linear functions of the coordinates; we have thence for (x, y, z, w) the three-fold relation

$$\left\| \begin{matrix} \mathrm{L}, & \delta_x \mathrm{U}_1, & \delta_x \mathrm{U}_2, & \delta_x \mathrm{U}_3, & \delta_x \mathrm{U}_4, & \delta_x \mathrm{U}_5 \\ \mathrm{M}, & \delta_y \mathrm{U}_1 & . & . & & \\ \mathrm{N}, & \delta_z \mathrm{U}_1 & . & & & \\ \mathrm{P}, & \delta_w \mathrm{U}_1 & & & & \end{matrix} \right\| = 0,$$

which denotes a system of $\frac{1}{6}\ 6 \cdot 5 \cdot 4 = 20$ points.

It would seem that if the line cuts in involution 6 given quadrics, there should be between the 6 coordinates a four-fold relation of the order $\frac{1}{2}10 = 5$; this would imply a reduction 25, viz. we should have $5 = 3 \cdot 10 - 25$. I do not understand this, and I drop the question.

99. I return to the question to find the relation between the coordinates (a, b, c, f, g, h) of a line which cuts in involution the 3 quadric surfaces

$$(a_1, b_1, c_1, d_1, f_1, g_1, h_1, l_1, m_1, n_1 \Sigma x, y, z, w)^2 = 0, \quad (a_2, \ldots \Sigma x, y, z, w)^2 = 0,$$
$$(a_3 \ldots \Sigma x, y, z, w)^2 = 0.$$

Writing down any two of the equations of the line, for instance

$$\begin{aligned} \mathrm{h}y - \mathrm{g}z + \mathrm{a}w &= 0, \\ -\mathrm{h}x \quad + \mathrm{f}z + \mathrm{b}w &= 0, \end{aligned}$$

if we substitute the values of (x, y) in the equation of the first surface,

it becomes $(a_1, \ldots \mathcal{J} fz+bw,\ gz-aw,\ hz,\ hw)^2 = 0$;
or if we write for shortness

$$\Pi = (f,\ g,\ h,\ o),$$
$$\Pi' = (b,\ -a,\ o,\ h),$$

then the equation is

$$(a_1, \ldots \mathcal{J}\Pi)^2 . z^2 + 2(a_1, \ldots \mathcal{J}\Pi\mathcal{J}\Pi') . zw + (a_1, \ldots \mathcal{J}\Pi')^2 . w^2 = 0,$$

and forming the like equations for the other two surfaces, the condition of involution is at once found to be

$$\begin{vmatrix} (a_1, \ldots \mathcal{J}\Pi)^2, & (a_1, \ldots \mathcal{J}\Pi\mathcal{J}\Pi'), & (a_1, \ldots \mathcal{J}\Pi')^2 \\ (a_2, \ldots \mathcal{J}\Pi)^2, & (a_2, \ldots \mathcal{J}\Pi\mathcal{J}\Pi'), & (a_2, \ldots \mathcal{J}\Pi')^2 \\ (a_3, \ldots \mathcal{J}\Pi)^2, & (a_3, \ldots \mathcal{J}\Pi\mathcal{J}\Pi'), & (a_3, \ldots \mathcal{J}\Pi')^2 \end{vmatrix} = 0.$$

100. It is convenient, in working this out, to consider Π, Π' as standing, in the first instance, for (x, y, z, w), (x', y', z', w'), these symbols being ultimately replaced by the above-mentioned values. Writing also, for shortness, (abc) to denote the determinant $a_1(b_2c_3 - b_3c_2) + \&c.$, and so in other cases, it is at once seen that the function on the right-hand side is a sum of such determinants each into a proper factor, containing the coordinates (a, b, c, f, g, h), originally in the order 6, but where each term contains the factor h^3, which may be omitted; or finally the result is of the order 3 in the coordinates. Thus we have a term

$$(abc) \begin{vmatrix} x^2, & xx', & x'^2 \\ y^2, & yy', & y'^2 \\ z^2, & zz', & z'^2 \end{vmatrix}$$

where the second factor is

$$x^2y'z'\,(yz' - y'z) + y^2z'x'\,(zx' - z'x) + z^2x'y'\,(xy' - x'y),\ = z^2x'y'\,(xy' - x'y),$$
$$= h^2\,(-ab)\,(-af - bg),\ = -abc\,h^3,$$

or, omitting the factor $-h^3$, the term is $(abc)\,abc$.

101. There are in all 120 terms, but 16 of these are found to vanish (viz., these are the terms in *agh, bhf, cfg; ahl, bfm, cgn; agl, bhm, cfn; dmn, dnl, dlm; fgn, ghl, hfm*). The final result contains therefore 104 terms; viz., as a further abbreviation writing *abc* &c., instead of (abc) &c., to denote the above-mentioned determinants, the equation is

$$\begin{aligned} &abc\,.\,abc - bcd\,.\,agh - cad\,.\,bhf - abd\,.\,cfg \\ &+ bcf\,.\,a^3 + cag\,.\,b^3 + abh\,.\,c^3 + adl\,.\,f^3 + bdm\,.\,g^3 + cdn\,.\,h^3 \\ &+ abn\,.\,c\,(bg - af) + adf\,.\,f\,(ch - bg) \\ &+ bcl\,.\,a\,(ch - bg) + bdg\,.\,g\,(af - ch) \\ &+ cam\,.\,b\,(af - ch) + cdh\,.\,h\,(bg - af) \\ &- bcg\,.\,a^2b - bch\,.\,a^2c + bcm\,.\,a^2g - bcn\,.\,a^2h \\ &- cah\,.\,b^2c - caf\,.\,b^2a + can\,.\,b^2h - cal\,.\,b^2f \\ &- abf\,.\,c^2a - abg\,.\,c^2b + abl\,.\,c^2f - abm\,.\,c^2g \end{aligned}$$

$$
\begin{array}{l}
\quad -adg\,.\,\mathrm{bf}^2 + adh\,.\,\mathrm{cf}^2 + adm\,.\,\mathrm{f}^2\mathrm{g} + adn\,.\,\mathrm{f}^2\mathrm{h} \\
\quad -bdh\,.\,\mathrm{cg}^2 + bdf\,.\,\mathrm{ag}^2 + bdn\,.\,\mathrm{g}^2\mathrm{h} + bdl\,.\,\mathrm{g}^2\mathrm{f} \\
\quad -cdf\,.\,\mathrm{ah}^2 + cdg\,.\,\mathrm{bh}^2 + cdl\,.\,\mathrm{h}^2\mathrm{f} + cdm\,.\,\mathrm{h}^2\mathrm{g} \\
+2\left\{\begin{array}{l} afg\,.\,\mathrm{b}^2\mathrm{c} - afh\,.\,\mathrm{bc}^2 + afl\,.\,\mathrm{bcf} - afm\,.\,\mathrm{c}^2\mathrm{h} - afn\,.\,\mathrm{b}^2\mathrm{g} \\ +bgh\,.\,\mathrm{c}^2\mathrm{a} - bgf\,.\,\mathrm{ca}^2 + bgm\,.\,\mathrm{cag} - bgl\,.\,\mathrm{a}^2\mathrm{f} - bgl\,.\,\mathrm{c}^2\mathrm{h} \\ +chf\,.\,\mathrm{a}^2\mathrm{b} - chg\,.\,\mathrm{ab}^2 + chn\,.\,\mathrm{abh} - chm\,.\,\mathrm{b}^2\mathrm{g} - chm\,.\,\mathrm{a}^2\mathrm{f} \end{array}\right\} \\
+2\left\{\begin{array}{l} agm\,.\,\mathrm{bcf} - agn\,.\,\mathrm{b}^2\mathrm{f} - ahm\,.\,\mathrm{c}^2\mathrm{f} + ahn\,.\,\mathrm{bcf} \\ +bhn\,.\,\mathrm{cag} - bfl\,.\,\mathrm{c}^2\mathrm{g} - bfn\,.\,\mathrm{a}^2\mathrm{g} + bfl\,.\,\mathrm{cag} \\ +cfl\,.\,\mathrm{abh} - chm\,.\,\mathrm{a}^2\mathrm{h} - cgl\,.\,\mathrm{b}^2\mathrm{h} + cgm\,.\,\mathrm{abh} \end{array}\right\} \\
+2\left\{\begin{array}{l} -amn\,.\,\mathrm{af}^2 - anl\,.\,\mathrm{bf}^2 - alm\,.\,\mathrm{cf}^2 + dfg\,.\,\mathrm{ch}^2 \\ -bnl\,.\,\mathrm{bg}^2 - blm\,.\,\mathrm{cg}^2 - bmn\,.\,\mathrm{ag}^2 + dgh\,.\,\mathrm{af}^2 \\ -clm\,.\,\mathrm{ch}^2 - cmn\,.\,\mathrm{ah}^2 - cnl\,.\,\mathrm{bh}^2 + dhf\,.\,\mathrm{bg}^2 \end{array}\right\} \\
+2\left\{\begin{array}{l} -dfl\,.\,\mathrm{fgh} - dfm\,.\,\mathrm{g}^2\mathrm{h} - dfn\,.\,\mathrm{gh}^2 \\ -dgm\,.\,\mathrm{fgh} - dgn\,.\,\mathrm{h}^2\mathrm{f} - dgl\,.\,\mathrm{hf}^2 \\ -dhn\,.\,\mathrm{fgh} - dhl\,.\,\mathrm{f}^2\mathrm{g} - dhm\,.\,\mathrm{fg}^2 \end{array}\right\} \\
-4fgh\,.\,\mathrm{abc} \\
+4\left\{\begin{array}{l} fgh\,.\,\mathrm{bch} - fgm\,.\,\mathrm{ach} - fmn\,.\,\mathrm{agh} - fnl\,.\,\mathrm{bgh} - flm\,.\,\mathrm{cgh} \\ +glm\,.\,\mathrm{caf} - gln\,.\,\mathrm{baf} - gnl\,.\,\mathrm{bhf} - glm\,.\,\mathrm{chf} - gmn\,.\,\mathrm{ahf} \\ +hfn\,.\,\mathrm{abg} - hfl\,.\,\mathrm{cbg} - hlm\,.\,\mathrm{cfg} - hmn\,.\,\mathrm{afg} - hnl\,.\,\mathrm{bfg} \end{array}\right\} = 0.
\end{array}
$$

And observe, by what precedes, this triple system of lines contains each of the following double systems: viz., the lines which meet the quadriquadric curve (2, 3) twice, those which meet the curve (3, 1) twice, those which meet the curve (1, 2) twice.

Persymmetrical Case: the Hessian of a Cubic.

102. Reverting to the general equation

$$\alpha\mathrm{S} + \beta\mathrm{T} + \gamma\mathrm{U} + \delta\mathrm{V} = \text{cone},$$

which connects the symmetroid and Jacobian, it is evident that if S, T, U, V are the derivatives, in regard to the coordinates, of a single cubic function U, $= (*\,\mathfrak{x}, y, z, w)^3$, then the symmetroid and the Jacobian become one and the same surface; viz., this is the Hessian surface H=0 derived from the given cubic surface. The two corresponding points on the symmetroid and the Jacobian respectively, and the two corresponding points on the Jacobian, become one and the same pair of corresponding points on the Hessian; viz., either of these points is such that its first polar surface in regard to the cubic is a quadric cone having for its vertex the other corresponding point. And the Hessian surface unites the properties of the Jacobian and the symmetroid, viz. it has 10 nodes and 10 lines. It is, in fact, known that there are five planes such that the intersection of every two of them is a line on the Hessian surface, and the intersection of every three of

them a node on the surface; viz., if the equations of the five planes are $x=0$, $y=0$, $z=0$, $w=0$, $u=0$, then the equation of the Hessian surface is

$$xyzwu\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}+\frac{d}{w}+\frac{e}{u}\right)=0,$$

a form which puts in evidence the properties just referred to.

Quartics with 11 or more Nodes.

103. I mention two results which, although they relate to quadric surfaces with more than 10 nodes, present themselves in such immediate connexion with the present Memoir, that it is natural to speak of them. If, in the equation

$$\begin{vmatrix} A, & H, & G, & L \\ H, & B, & F, & M \\ G, & F, & C, & N \\ L, & M, & N, & D \end{vmatrix} = 0$$

of the symmetroid (A, B, .. linear functions of the coordinates), we have identically $A=0$, then the surface has evidently a node $H=0$, $G=0$, $L=0$; viz., this is a node in addition to the usual 10 nodes, or the surface has in all 11 nodes. And so also if (identically in every case) B is $=0$, there are 12 nodes; if C is $=0$, there are 13 nodes, and if D is $=0$, there are 14 nodes. These are, in fact, quartic surfaces with 11, 12, 13, and 14 nodes respectively, mentioned in Kummer's Memoir.

104. We may consider the symmetroid derived from the quadric surfaces which pass through 6 given points; viz., taking as before (see No. 25) the coordinates of the 6 points to be $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(1, 1, 1, 1)$, $(\alpha, \beta, \gamma, \delta)$, and (a, b, c, f, g, h) as the coordinates of the line joining the last-mentioned two points, and, to avoid confusion, taking for the present purpose (X, Y, Z, W) instead of $(\alpha, \beta, \gamma, \delta)$ for the coordinates of a point on the symmetroid, the equation is obtained by arranging in the form of a determinant the coefficients of the quadric form

$$\begin{aligned} &Xx\,(\quad\quad hy-gz+\alpha w) \\ &+Yy\,(-hx\quad\quad +fz+bw) \\ &+Zz\,(\quad gx-fy\quad\quad +cw) \\ &+W\,(\alpha x+\beta y+\gamma z+\delta w)\,; \end{aligned}$$

viz., the equation in question is

$$\begin{vmatrix} . & . & h\overline{X-Y}+cW, & g\overline{Z-X}+bW, & aX \\ h\overline{X-Y}+cW, & . & . & f\overline{Y-Z}+aW, & bY \\ g\overline{Z-X}+bW, & f\overline{Y-Z}+aW, & . & . & cZ \\ aX, & bY, & cZ, & . & . \end{vmatrix} = 0\,;$$

or, as it may be more simply written,

$$\sqrt{aX}\{f(Y-Z)+\alpha\alpha W\}+\sqrt{bY}\{g(Z-X)+b\beta W\}$$
$$+\sqrt{cZ}\{h(X-Y)+c\gamma W\}=0.$$

This is, in fact, a surface with 16 nodes. It would appear additional nodes correspond to the six common intersections of the quadric surfaces, or nodes of the Jacobian; and it would seem that for four quadric surfaces having in common 1, 2, 3, 4, 5, or 6 points, the corresponding symmetroid would have 11, 12, 13, 14, 15, or 16 nodes. But I reserve this for future consideration.

I take the opportunity of mentioning some results which have a connexion, although not an immediate one, with the subject of the present Memoir.

Quadric Surface through three given Lines.

105. To find the equation to the quadric surface through the three lines $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, b_2, c_2, f_2, g_2, h_2)$, $(a_3, b_3, c_3, f_3, g_3, h_3)$. Take on one of the lines the points $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$; then the equation of a quadric surface through this line will be of the form

$$\begin{vmatrix} x^2 & y^2 & z^2 & w^2 & yz & zx & xy & xw & yw & zw \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 & \beta\gamma & \gamma\alpha & \alpha\beta & \alpha\delta & \beta\delta & \gamma\delta \\ 2\alpha\alpha' & 2\beta\beta' & 2\gamma\gamma' & 2\delta\delta' & \beta\gamma'+\beta'\gamma & \gamma\alpha'+\gamma'\alpha & \alpha\beta'+\alpha'\beta & \alpha\delta'+\alpha'\delta & \beta\delta'+\beta'\delta & \gamma\delta'+\gamma'\delta \\ \alpha'^2 & \beta'^2 & \gamma'^2 & \delta'^2 & \beta'\gamma' & \gamma'\alpha' & \alpha'\beta' & \alpha'\delta' & \beta'\delta' & \gamma'\delta' \\ \vdots & & & & & & & & & \end{vmatrix} = 0;$$

and if we form thus a determinant with three of its lines relating to the line 1, three of them to the line 2, and three to the line 3, we have the equation of the quadric surface through the three lines. But considering in the determinant the three lines which refer to the line 1, it is clear that the determinant is a function of the order 3 of the coordinates $(a_1, b_1, c_1, f_1, g_1, h_1)$ of the line in question; and the like as regards the other two lines respectively. Now observe that if two of the lines intersect, the problem becomes indeterminate (in fact, the plane of the intersecting lines, and any plane whatever through the third line, constitute a solution); the condition for the intersection of the lines 1 and 2 is $a_1f_2+a_2f_1+b_1g_2+b_2g_1+c_2h_1+c_1h_2=0$; hence, if this condition be satisfied, the determinant must vanish; it therefore divides by the factor a_1f_2+&c.; but, similarly, it divides by the factors a_2f_3+&c. and a_3f_1+&c.; and throwing out the three factors, the result should be of the order 1, that is linear, in regard to the three sets of coordinates respectively. I have obtained this reduced result in my "Memoir on the Six Coordinates of a Line" (*Camb. Phil. Trans.*, t. xi., 1869,

p. 311); viz., writing (abc) to denote the determinant $a_1(b_2c_3-b_3c_2)+\&c.$, and so for the other like determinants, the result is

$$\begin{aligned}&(agh)\,x^2+(bhf)\,y^2+(cfg)\,z^2+(abc)\,w^2\\+&[(abg)-(cah)]\,xw+[(bfg)+(chf)]\,yz\\+&[(bch)-(abf)]\,yw+[(cgh)+(afg)]\,zx\\+&[(caf)-(bcg)]\,zw+[(ahf)+(bgh)]\,xy=0.\end{aligned}$$

Condition that five given lines may lie in a Cubic Surface.

106. Taking the lines to be $(a_1, b_1, c_1, f_1, g_1, h_1)$, ... $(a_5, b_5, c_5, f_5, g_5, h_5)$, and $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ the coordinates of any two points on one of the lines, the equation of a cubic surface through this line would be

$$\left|\begin{array}{lllll} x^3 & \ldots\ldots & x^2y & \ldots\ldots & xyz \qquad \ldots\ldots \\ \alpha^3 & & \alpha^2\beta & & \alpha\beta\gamma \\ 3\alpha^2\alpha' & & 2\alpha\alpha'\beta+\alpha^2\beta' & & \alpha'\beta\gamma+\alpha\beta'\gamma+\alpha\beta\gamma' \\ 3\alpha\alpha'^2 & & 2\alpha\alpha'\beta'+\alpha'^2\beta & & \alpha\beta'\gamma'+\alpha'\beta\gamma'+\alpha\beta'\gamma' \\ \alpha'^3 & & \alpha'^2\beta' & & \alpha'\beta'\gamma' \end{array}\right| = 0;$$

and hence it at once appears that, forming a determinant of 20 lines, wherein four lines relate to the line 1, four to the line 2, ..., four to the line 5, and equating this to zero, we have the required condition. But the condition so obtained is of the order $(\tfrac{1}{2}4\,.\,3=)$ 6 in regard to the coordinates of each line; and, as for the quadric, it is satisfied identically if we have any such equation as $a_1f_2+\&c.=0$; it consequently contains the several factors $a_1f_2+\&c.$, which can be formed with the coordinates of any two of the five lines; and throwing out these factors, the condition should be of the order 2 in regard to the coordinates of each line. We in fact know that the required relation between the five lines is that they shall all of them be cut by a sixth line; and moreover that, writing $a_1f_2+a_2f_1+b_1g_2+b_2g_1+c_1h_2+c_2h_1=12$, &c., then that the condition for this is

$$\left|\begin{array}{lllll} . & 12, & 13, & 14, & 15 \\ 21, & . & 23, & 24, & 25 \\ 31, & 32, & . & 34, & 35 \\ 41, & 42, & 43, & . & 45 \\ 51, & 52, & 53, & 54, & . \end{array}\right| = 0,$$

being, as it should be, of the order 2 in regard to the coordinates of each line.

Condition that 7 given lines shall lie on a Quartic Surface.

107. Taking the lines to be $(a_1, b_1, c_1, f_1, g_1, h_1)$ ... $(a_7, b_7, c_7, f_7, g_7, h_7)$, then in precisely the same way we form a determinant of the order

$(\frac{1}{2}5.4 =)\ 10$ in regard to the coordinates of each line; this determinant however divides out by the several factors $a_1 f_2 +$ &c., which can be formed with the seven lines; or throwing these out and equating the quotient to zero, we have an equation of the order 4 in regard to the coordinates of each line. It would not be practicable to obtain the reduced equation in this manner, and I do not know how to obtain it otherwise, but the material conclusion is that the order is $= 4$.

The Jacobian of 6 *points.*

108. Any 6 points whatever may be regarded as points on a skew cubic; and the coordinates (x, y, z, w) may be taken so that the equations of the skew cubic shall be $\left\| \begin{matrix} x, y, z, \\ y, z, w \end{matrix} \right\| = 0$. This being so, the coordinates of the 6 given points may be taken to be $(1, t_1, t_1^2, t_1^3) \ldots (1, t_6, t_6^2, t_6^3)$; and the equation of the Jacobian surface of the 6 points can then be expressed in a very simple form, putting in evidence the passage of the surface through the skew cubic; viz. writing $p_1 = \Sigma t_1,\ p_2 = \Sigma t_1 t_2,\ p_3 = \Sigma t_1 t_2 t_3,\ p_4 = \Sigma t_1 t_2 t_3 t_4,\ p_5 = \Sigma t_1 t_2 t_3 t_4 t_5,\ p_6 = t_1 t_2 t_3 t_4 t_5 t_6$; moreover,

$$\square = \tfrac{1}{2}(6xyzw - 4xz^3 - 4y^3w + 3y^2z^2 - x^2w^2),$$

and therefore

$$\begin{aligned} \delta_x \square &= -xw^2 - 2z^3 + 3yzw \\ \delta_y \square &= 3yz^2 - 6y^2w + 3xzw \\ \delta_z \square &= 3y^2z - 6xz^2 + 3xyz \\ \delta_w \square &= -x^2w - 2y^3 + 3xyz; \end{aligned}$$

then the equation of the Jacobian surface is

$$\begin{aligned} & 3\,(\ xp_3 \qquad + zp_1 - 2w\ )\ \delta_x \square \\ + & \ (\qquad\qquad 2zp_2 - wp_1)\ \delta_y \square \\ + & \ (\ xp_5 - 2yp_4 \qquad\ )\ \delta_z \square \\ + 3 & \ (2xp_6 - \ yp_5 \qquad - wp_3)\ \delta_w \square = 0. \end{aligned}$$

There is not much difficulty in the direct investigation; but a simple verification may be obtained by showing that the surface contains upon it the 15 lines 12, 13, ... 56. Write in the equation

$$(x, y, z, w) = (\lambda + \mu,\ \lambda s + \mu t,\ \lambda s^2 + \mu t^2,\ \lambda s^3 + \mu t^3),$$

the values $\delta_x \square$ &c. are found to contain the factor $\lambda\mu\,(s-t)^3$, and omitting this common factor the values are as

$$\tfrac{1}{3}(\lambda s^3 - \mu t^3),\ -(\lambda s^2 - \mu t^2),\ (\lambda s - \mu t),\ -\tfrac{1}{3}(\lambda - \mu);$$

the equation thus becomes

$$\begin{aligned} & \{\lambda\,(-2s^3 + s^2p_1 + p_3) + \mu\,(-2t^3 + t^2p_1 + p_3)\}\ (\lambda s^3 - \mu t^3) \\ - & \{\lambda\,(-s^3p_1 + 2s^2p_2 \qquad) + \mu\,(-t^3p_1 + 2t^2p_2 \qquad)\}\ (\lambda s^2 - \mu t^2) \\ + & \{\lambda\,(-2sp_4 + p_5 \qquad) + \mu\,(-2tp_4 + p_5 \qquad)\}\ (\lambda s - \mu t) \\ - & \{\lambda\,(-s^3p_3 - sp_5 + 2p_6) + \mu\,(-t^3p^3 - tp_5 + 2p_6)\}\ (\lambda - \mu) = 0, \end{aligned}$$

viz., collecting the terms, the coefficient of λ_{ss} vanishes, and the whole is

$$-2\lambda^3(1, p_{12}p_{34}, p_{13}p_{42}, p_{14}p_{23}, p_{15}p_{26}, p_{16}p_{25}\Sigma\theta, -1)^\theta$$
$$+2\mu^3(1, p_{12}p_{34}, p_{13}p_{42}, p_{14}p_{23}, p_{15}p_{26}, p_{16}p_{25}\Sigma\phi, -1)^\theta = 0;$$

viz., this equation is satisfied if θ denote any one of the quantities $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$, and ϕ any one of the same 6 quantities, that is, the equation of the surface is satisfied when (x, y, z, w) are the coordinates of a point on the line joining any 2 of the 6 points.

Locus of the vertex of a Quadric Cone which touches each of Six given Lines.

109. Representing as before each line by means of its six coordinates, let $(\alpha, \beta, \gamma, \ldots)$ be the coordinates of the vertex, and (X, Y, Z, W) current coordinates. Suppose that (a, b, c, f, g, h) are the coordinates of any one of the lines, the equation of the plane through this line and the vertex is

$$a(\alpha W - wX) + b(\beta W - wY) + c(\gamma W - wZ)$$
$$+ f(\beta Z - \gamma Y) + g(\gamma X - \alpha Z) + h(\alpha Y - \beta X) = 0;$$

or what is the same thing, writing for shortness

$$\begin{aligned} P &= \quad . \quad hy - gz + aw \\ Q &= -hx \quad . + fz + bw \\ R &= \quad gx - fy \quad . + cw \\ S &= -ax - by - cz \quad . \end{aligned}$$

the equation is $PX + QY + RZ + SW = 0$.

The plane in question is a tangent plane to the cone touched by the 6 lines. Now when 6 planes touch a quadric cone, their traces on any plane whatever touch a conic the intersection of the cone by that plane. Hence taking the plane $W = 0$, the equation of the trace is

$$PX + QY + RZ = 0,$$

and forming in like manner the equations belonging to each of the given lines, the condition that the 6 traces may touch a conic is

$$(P^2, Q^2, R^2, QR, RP, PQ) = 0,$$

where the left hand side represents a determinant of 6 lines, the several lines being respectively $P_1^2, Q_1^2, R_1^2, Q_1R_1, R_1P_1, P_1Q_1$; P_2^2, &c. Or more simply we may denote the equation by

$$[(P, Q, R)^2] = 0.$$

To ascertain the form of this, write for a moment $y = 0$, $z = 0$; the equation is $[(aw, -hx + bw, gx + cw)^2] = 0$,

or attending only to the highest and lowest powers of w, this is

$$w^{12}[(a, b, c)^2] \ldots + w^4x^8[(a, -h, g)^2] = 0;$$

and it is thence easy to infer that the whole equation divides by w^4.

so that, omitting this factor, the form of the equation is

$$((a, b, c, f, g, h)^2 \mathfrak{Z} x, y, z, w)^3 = 0;$$

viz., the equation is of the order 3 in the coordinates (x, y, z, w), and of the degree 2 in the coordinates (a, b, c, f, g, h) of each of the lines. It would not be very difficult to actually develope the equation; in fact, starting from the term $w^3 \{(a, b, c)^2\}$ the other terms are obtained therefrom by changing a, b, c into $a + \frac{1}{w}(hy - gz)$, $b + \frac{1}{w}(-hx + fz)$, $c + \frac{1}{w}(gx - fy)$ respectively; the equation may therefore be written in the symbolic form

$$w^3 . \operatorname{exp}. \frac{1}{w} \{(hy - gz)\partial_a + (-hx + fz)\partial_b + (gx - fy)\partial_c\} . \{(a, b, c)^2\} = 0,$$

or what is the same thing

$$w^3 . \operatorname{exp}. \frac{1}{w} \{x(g\partial_c - h\partial_b) + y(h\partial_a - f\partial_c) + z(f\partial_b - g\partial_a)\} . \{(a, b, c)^2\} = 0,$$

where exp. 0 (read exponential) denotes e^0, and $\{(a, b, c)^2\}$ represents a determinant as above explained. The equation contains, it is clear, the four terms

$$x^3 \{(a, -h, g)^2\} + y^3 \{(-h, b, -f)^2\} + z^3 \{(g, f, c)^2\} + w^3 \{(a, b, c)^2\}.$$

I am not sure whether this surface of the eighth order has been anywhere considered.

Mr. J. J. Walker next communicated a Note on his paper read at the January Meeting of the Society.

On Conditions for, and Equations of, Corresponding Points in certain Involutions. By J. J. Walker.

The principal points discussed in the following short paper are:

1. The condition to be satisfied in order that a given binary quadric should determine two corresponding points in one of the three involutions determined by a quartic; from which condition is deduced the cubic giving the three points corresponding to an assigned one in each of those involutions respectively, the sextic determining their double points, and the cubic of centres:

2. The corresponding questions when the three involutions are determined by two quadrics instead of one quartic—any previously published discussion of which case, as far as I am aware, having been confined to the particular instance in which the roots of each quadric correspond, leaving unconsidered those in which a root of one corresponds to a root of the other:

3. The condition to be satisfied in order that two binary cubics should determine an involution in such a manner that each root of one corresponds to a root of the other; and, when this condition is fulfilled, the equations which give the point corresponding to any seventh assigned one in the involution, the double points or foci, and the centre. The involution condition is found under two distinct forms, in one of which it is expressed as a function of the coefficients of the quadric and quartic covariants of the two cubics; the particular case of one cubic being the cubic covariant of the other is treated of, and the application of the involution condition to some questions in connexion with the theory of cubic curves considered.

The questions, as thus enumerated, are discussed both by the application of elementary principles only, and likewise, where the results are invariant, by the principles of the Higher Algebra, which afford so much readier demonstrations.

4. It will be convenient to premise, for subsequent reference, a few elementary principles. If the distances of four points p, q, r, s, from any origin o be given explicitly equal to α, β, γ, δ, and h_1, h_2, h_3 be the distances from origin of the centres of the three involutions determined by these four points, in which pq, pr, ps are respectively corresponding points, while $h_1 \pm k_1$, $h_2 \pm k_2$, $h_3 \pm k_3$ are the distances of the pairs of double points respectively from origin; then those of any two corresponding points will be connected by the relation

$$x'x - h(x' + x) + h^2 - k^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

suffixes being written to indicate in which of the three systems the points are supposed to stand; and, in particular,

$$\alpha\beta - h_1(\alpha + \beta) + h_1^2 - k_1^2 = 0,$$
$$\gamma\delta - h_1(\gamma + \delta) + h_1^2 - k_1^2 = 0,$$

from which equations

$$h_1 = \frac{\alpha\beta - \gamma\delta}{\alpha + \beta - \gamma - \delta} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

$$h_1^2 - k_1^2 = \frac{\alpha\beta(\gamma + \delta) - \gamma\delta(\alpha + \beta)}{\alpha + \beta - \gamma - \delta} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3);$$

and corresponding values for h_2, $h_2^2 - k_2^2$ are obtained by interchanging β and γ; for h_3 and $h_3^2 - k_3^2$ by interchanging β and δ, in (2) and (3).

Again, if p', q', r', s' be the inverse points, with respect to o, of p, q, r, s, so that their distances from o are to be represented by $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $\frac{1}{\gamma}$, $\frac{1}{\delta}$ respectively; and if h_1', h_2', h_3', $h_1' \pm k_1'$, ... represent the distances from o of the centres and double points in the involutions in which $p'q'$, $p'r'$, $p's'$ are respectively corresponding points,

$$h_1' = \frac{\alpha\beta - \gamma\delta}{\alpha\beta(\gamma+\delta) - \gamma\delta(\alpha+\beta)} = \frac{h_1}{h_1^2 - k_1^2} \quad \ldots\ldots\ldots\ldots (4),$$

$$h_1'^2 - k_1'^2 = \frac{\alpha+\beta-(\gamma+\delta)}{\alpha\beta(\gamma+\delta) - \gamma\delta(\alpha+\beta)} = \frac{1}{h_1^2 - k_1^2} \quad \ldots\ldots\ldots\ldots (5).$$

5. If the distances of the points p, q, r, s from o be supposed not given explicitly, but that α, β, γ, δ are roots of a quartic

$$(abcde\,\mathcal{X}1)^4 = 0 \quad \ldots\ldots\ldots\ldots (6),$$

then the three points which correspond to an assigned one, whose distance from o is x', in the three involutions respectively determined by the points p, q, r, s, will be given implicitly by a cubic, which I proceed to form.

Substituting in (1) the values of h and $h^2 - k^2$ from (2), (3),

$$(\alpha+\beta-\gamma-\delta)\,x'x - (\alpha\beta-\gamma\delta)(x'+x) + \alpha\beta(\gamma+\delta) - \gamma\delta(\alpha+\beta) = 0$$

determines the point corresponding to that whose distance from o is x' in the first involution. Hence, if for shortness we write

$$l_1 = \alpha+\beta-\gamma-\delta,$$
$$m_1 = \alpha\beta-\gamma\delta,$$
$$n_1 = \alpha\beta(\gamma+\delta) - \gamma\delta(\alpha+\beta),$$

and l_2, m_2, n_2 represent what l_1, m_1, n_1 respectively become when β and γ are interchanged; l_3, m_3, n_3 what the same expressions become when β and δ are interchanged, the cubic sought will be

$$\{l_1x'x - m_1(x'+x) + n\}\{l_2x'x - m_2(x'+x) + n_2\}\{l_3x'x - m_3(x'+x) + n_3\} = 0,$$

or
$$l_1l_2l_3 \,.\, x'^3x^3 - \Sigma l_1l_2m_3 \,.\, x'^2x^2(x'+x) + \Sigma l_1m_2m_3 \,.\, x'x(x'+x)^2 + \Sigma l_1l_2n_3 \,.\, x'^2x^2$$
$$- m_1m_2m_3(x'+x)^3 - \Sigma l_1m_2n_3 \,.\, x'x(x'+x) + \Sigma m_1m_2n_3 \,.\, (x'+x)^2$$
$$+ \Sigma l_1n_2n_3 \,.\, x'x - \Sigma m_1n_2n_3(x'+x) + n_1n_2n_3 = 0 \quad \ldots\ldots (7).$$

6. Let

$$\mathrm{G} = a^2d - 3abc + 2b^3,$$
$$6\mathrm{F} = a^2e + 2abd - 9ac^2 + 6b^2c,$$
$$3\mathrm{E} = abe - 3acd + 2b^2d,$$
$$2\mathrm{D} = b^2e - ad^2,$$

and G′, F′, E′, D′ represent what G, F, E, D respectively become when a and e, b and d are interchanged; then it will be found that the symmetric functions of the roots of the quartic, which enter into the above equation, are expressed in terms of G, F .... D′ thus:

$$l_1l_2l_3 = -32\mathrm{G},$$
$$\Sigma l_1l_2m_3 = 96\mathrm{F},$$
$$\Sigma l_1l_2n_3 = \Sigma l_1m_2m_3 = -96\mathrm{E},$$
$$m_1m_2m_3 = 32\mathrm{D},$$
$$\Sigma l_1m_2n_3 = 192\mathrm{D},$$
$$\Sigma l_1n_2n_3 = \Sigma m_1m_2n_3 = 96\mathrm{E}',$$
$$\Sigma m_1n_2n_3 = -96\mathrm{F}',$$
$$n_1n_2n_3 = 32\mathrm{G}'.$$

Substituting in (7) and dividing by -32, it becomes

$$Gx'^3x^3+3Fx'^2x^2(x'+x)+3E\{(x'+x)^2+x'x\}x'x+D\{(x+x')^2+6x'x\}(x'+x)$$
$$-3E'\{(x'+x)^2+x'x\}-3F'(x'+x)-G'=0 \ldots (8),$$

which may be re-arranged, replacing x' by $\frac{x'}{y'}$ and x by $\frac{x}{y}$,

$$0=\big((G,F,E,D\mathcal{X}x'y')^3,\ (F,E,D,-E'\mathcal{X}x'y')^3,\ (E,D,-E',-F'\mathcal{X}x'y')^3,$$
$$(D,-E',-F',-G'\mathcal{X}x'y')\mathcal{X}xy\big)^3 \ldots (9),$$

which is the cubic it was proposed to investigate, and of which a quite different demonstration will be given after noticing two special cases.

7. To obtain the equation which determines the three centres of the involutions, it is only necessary to divide (8) by x'^3, and then suppose it to become infinitely great: thus there results, replacing x by $\frac{x}{y}$,

$$(G,F,E,D\mathcal{X}xy)^3=0 \ldots\ldots\ldots\ldots\ldots (10),$$

the roots of which, expressed in terms of those of the given quartic, are

$$h_1=\frac{\alpha\beta-\gamma\delta}{\alpha+\beta-\gamma-\delta},\quad h_2=\frac{\alpha\gamma-\beta\delta}{\alpha+\gamma-\beta-\delta},\quad h_3=\frac{\alpha\delta-\beta\gamma}{\alpha+\delta-\beta-\gamma}.$$

The above is a new "reducing" cubic, by the aid of which the quartic might be solved. Its connection with Lagrange's reducing cubic may be exhibited by transforming the roots. For, writing

$$\tfrac{1}{4}\{(\alpha+\beta)^2-(\alpha-\beta)^2\} \text{ for } \alpha\beta,\quad \tfrac{1}{4}\{(\gamma+\delta)^2-(\gamma-\delta)^2\} \text{ for } \gamma\delta,$$

$$4h_1=\frac{(\alpha+\beta)^2-(\gamma+\delta)^2+(\gamma-\delta)^2-(\alpha-\beta)^2}{\alpha+\beta-\gamma-\delta}$$
$$=\frac{(\alpha+\beta+\gamma+\delta)(\alpha+\beta-\gamma-\delta)+(\alpha+\gamma-\beta-\delta)(\beta+\gamma-\alpha-\delta)}{\alpha+\beta-\gamma-\delta}$$
$$=-\frac{4b}{a}-\frac{l_1l_2l_3}{l_1^2}=-\frac{4b}{a}+\frac{32G}{l_1^2},$$

or

$$8G\left(h_1+\frac{b}{a}\right)^{-1}=l_1^2;$$

but the roots of Lagrange's cubic are l_1^2, l_2^2, l_3^2.

8. Again, the double points, or foci, of the three involutions will be determined by making $x'=x$ in (8), which gives the sextic, replacing x by $\frac{x}{y}$,

$$(G,F,E,D,-E',-F',-G'\mathcal{X}xy)^6=0 \ldots\ldots\ldots (11).$$

This is the sextic covariant of the quartic, the geometrical significance of which has already been pointed out by Dr. Salmon (Higher Algebra, § 210, 2nd ed.)

9. Multiplying equation (8) by a'^3, then substituting c' for $a'x'x$, $-2b'$

for $a'(x'+x)$, the condition that $(a'b'c'\overline{\chi}xy)^2$ should determine two points, conjugate one to the other in one of the three involutions determined by the quartic $(abcde\overline{\chi}xy^4)$, is at once obtained; viz.,

$$Gc'^3-6Fb'c'^2+3E(4b'^2+a'c')c'-4D(2b'^2+3a'c')b' \\ -3E'a'(4b'^2+a'c')+6F'a'^2b-G'a'^3=0 \ldots (12).$$

This must be an invariant of the system of the quartic (u) and the quadratic (v), and the coefficients of each quantic enter in the third degree. I find that it is formed thus:—

The quartic covariant $\dfrac{du}{dx}\dfrac{dv}{dy}-\dfrac{dv}{dx}\dfrac{du}{dy}$ is

$$(ab'-a'b,\ ac'+2bb'-3a'c,\ 3(bc'-a'd),\ 3cc'-2b'd-a'e,\ c'd-b'e\overline{\chi}xy)^4;$$

and the cubic invariant of this covariant is (to a numerical factor) precisely the condition (12).

10. An *à posteriori* proof of the condition may now be readily obtained from the following considerations:—

It is well known that the Anharmonic Ratio of a pencil of lines is unaffected by any linear transformation of its equation; consequently, if the equation of six lines can be so transformed that the terms containing odd powers of the variables disappear, the pencil must be one of involution, since the rays of the transformed are equally inclined in pairs to the axis if these are rectangular, or in general intercept on any transversal parallel to one axis equal segments in pairs on either side of the point where the other axis meets the transversal.

Now, if the quartic and quadratic can be transformed simultaneously into $Ax^4+4Cx^2y^2+Ey^4$ and $A'x^2+C'y^2$, the cubic invariant of their quartic covariant above must vanish, since every term in (12) vanishes separately* on the suppositions $b=0$, $d=0$, $b'=0$. The vanishing of this invariant is therefore the condition of such a transformation being possible, *i. e.*, of the two forms determining six points, or lines, in involution.

The condition (12) having been established by this reasoning, by reversing the substitutions, *i. e.*, putting $a'x'x$ for c', and $-a'(x'+x)$ for $2b'$, the general equation (8) is arrived at.

11. The quadratic covariant $\overline{1.2}^2$ of uv is

$$(ac'+a'c-2bb')x^2+2(bc'+a'd-2b'c)xy+(cc'+a'e-2b'd)y^2 \ldots (13).$$

Now, when u, v are transformed so that the terms containing odd powers of x and y disappear, it is plain that the term xy in the above covariant also vanishes. Hence it follows that if v determine two corresponding points in one of the three involutions determined by u, then $\overline{1.2}^2$ will determine another pair of corresponding points in the

* For every term in D, E, G contains either b or d as a factor.

same involution. This consideration makes it possible to discriminate, as it were, that particular one of the three involutions to which the points determined by v belong. It is known that when two pairs of corresponding points are determined in an involution by two quadratics $(a'b'c' \overline{\times} xy)^2$, $(a''b''c'' \overline{\times} xy)^2$, then any other pair $(x'x)$ of corresponding points will satisfy

$$2(a'b''-a''b')x'x+(a'c''-a''c')(x'+x)+2(b'c''-b''c')=0 \ldots (14);$$

so that, if the covariant $\overline{1.2}^2$ (13) be written $(a''b''c'' \overline{\times} xy)$, the foci of the involution particularized will be determined by

$$(a'b''-a''b')x^2+(a'c''-a''c')xy+(b'c''-b''c')y^2=0 \ldots\ldots (15),$$

and the centre of the same involution by

$$2(a'b''-a''b')x+(a'c''-a''c')y=0 \ldots\ldots\ldots\ldots (16).$$

12. In the next place, the four points p, q, r, s may be supposed to be determined by two quadrics $u \equiv (abc \overline{\times} xy)^2$ and $v \equiv (a'b'c' \overline{\times} xy)^2$. But here one of the three involutions can be distinguished at once from the other two; viz., that in which one pair of corresponding points is determined by one quadric, the other pair by the other. This case has naturally attracted most attention, and has been fully discussed; but for completeness the results arrived at are here noticed and demonstrated, as they readily may be, in accordance with the method of proof above employed.

If $w \equiv (a''b''c'' \overline{\times} xy)^2$ is to determine a third pair of points, a certain condition must be satisfied identical with that for it being possible to transform u, v, w by the same substitutions into

$$Ax^2+Cy^2,\ A'x^2+C'y^2,\ A''x^2+C''y^2.$$

But for these forms the invariant

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ a'', & b'', & c'' \end{vmatrix}$$

evidently vanishes; this, then, or

$$a''(b'c-bc')+b''(ac'-a'c)+c''(a'b-ab')=0 \ldots\ldots (17),$$

is the required condition.

If, in (17), b'' be replaced by $-\frac{a''}{2}(x'+x)$, c'' by $a''x'x$, the equation connecting two corresponding points is obtained in the form used above (14); and the foci are determined by making $x'=x$, the centre by making x' infinite.

13. In the other two cases a root of one quadric corresponds to one of the other; consequently, the forms into which the quadrics should be changed to determine corresponding points equidistant from any origin, on either side of it, are

$$Ax^2+2Bxy+Cy^2 \quad \text{and} \quad kAx^2-2kBxy+kCy^2.$$

If a third quadratic $w \equiv (a''b''c'' \,\overline{|}\, xy)^2$ is to determine a pair of corresponding points in one of these two involutions, it must be possible, by the same substitutions as transform u, v into the forms above, to transform w into $A''x^2+C''y^2$.

Consider now the invariant of the system uvw

$$(ac''+a''c-2bb'')^2(a'c'-b'^2)-(a'c''+a''c'-2b'b'')^2(ac-b^2) \ldots(18).$$

This vanishes identically for the special forms

$$(A, B, C \,\overline{|}\, xy)^2, \quad (kA, -kB, kC \,\overline{|}\, xy)^2, \quad (A'', 0, C'' \,\overline{|}\, xy)^2;$$

and it is therefore the condition that $(a''b''c'' \,\overline{|}\, xy)^2$ should determine two corresponding points in one of those two involutions determined by $(abc \,\overline{|}\, xy)^2$ and $(a'b'c' \,\overline{|}\, xy)^2$, in which either root of the former corresponds to a root of the latter.*

Replacing $-\dfrac{2b''}{a''}$ by $x'+x$, and $\dfrac{c''}{a''}$ by $x'x$, in (18),

$$\{ax'x+b(x'+x)+c\}^2(a'c'-b'^2)-\{a'x'x+b'(x'+x)+c'\}^2(ac-b^2)=0 \ldots(19)$$

gives the two points corresponding to an assigned one, at a distance x' from origin, in the two involutions respectively.

Again, making $x'=x$, in (19),

$$(ax^2+2bx+c)^2(a'c'-b'^2)-(a'x^2+2b'x+c')^2(ac-b^2)=0 \ldots\ldots(20),$$

or the two quadrics

$$(a'c'-b'^2)^{\frac{1}{2}}(ax^2+2bxy+cy^2) \pm (ac-b^2)^{\frac{1}{2}}(a'x^2+2b'xy+c'y^2)=0,$$

determine the foci of the involutions; and, making x' infinite in (19), after division by x'^2,

$$(ax+by)^2(a'c'-b'^2)-(a'x+b'y)^2(ac-b^2) \ldots\ldots\ldots\ldots (21),$$

or

$$(a'c'-b'^2)^{\frac{1}{2}}(ax+by) \pm (ac-b^2)^{\frac{1}{2}}(a'x+b'y),$$

determines the two centres.

14. The equation (19) might have been obtained directly by forming the product of

$$(\alpha+\alpha'-\beta-\beta')x'x-(\alpha\alpha'-\beta\beta')(x'+x)+\alpha\alpha'(\beta+\beta')-\beta\beta'(\alpha+\alpha'),$$

and $(\alpha+\beta'-\alpha'-\beta)x'x-(\alpha\beta'-\alpha'\beta)(x'+x)+\alpha\beta'(\alpha'+\beta)-\alpha'\beta(\alpha+\beta')$,

* It is easily verified that this condition is satisfied when $(a''b''c'' \,\overline{|}\, xy)^2=0$ is the equation (15) determining the foci of the involution in which the roots of u correspond, also those of v; thus affording a proof of M. Chasles' theorem, that these foci are corresponding points in each of the other two involutions. Otherwise, this appears from the fact that (15)—which is a covariant of u, v—loses the term xy when u and v are transformed as above.

(α, β being supposed to be the roots of u; α', β' those of v,) and expressing the symmetric functions of the roots in terms of the coefficients; since this product must evidently vanish if x', x are to determine corresponding points in one of the two involutions. The result would be readily indentified with (19).

15. The case of an involution determined by two cubics, $u = (abcd\,\overline{\times}\,xy)^3$, $v = (a'b'c'd'\,\overline{\times}\,xy)^3$, in such a manner that the roots of one correspond each to one of those of the other, has never heretofore been considered, as far as I am aware; yet it is a case which frequently presents itself in the theory of cubic curves, particularly those of the third class. A certain condition must be fulfilled by the coefficients of $u\,v$, which I shall refer to as the "involution condition of a pair of cubics." It may be shown *à priori* that it must be of the sixth degree in the coefficients of each cubic, from the following considerations:—Let the roots of u be α, β, γ, those of v, α', β', γ'; and suppose α, β, α', β' to be given. These determine two involutions, in which α corresponds to α' or β'; consequently, if further γ be supposed to be given, and γ' is to be determined so as to correspond to it, the condition must be of the second degree in γ', *i. e.*, of the second degree in the coefficients of v, since these contain the root γ' implicitly in the first degree. But as there are three combinations of α, β, γ taken two and two, the general condition must be made up of the product of three such partial conditions, *i. e.*, be of the sixth degree in the coefficients of u; and since the same reasoning will apply where α', β', γ' are given, and α, β, γ successively to be determined from the other two of the three being supposed given, the coefficients of u must also enter in the sixth degree.

In the next place, it may be observed, that if two roots of u become equal, the condition should imply that two of v necessarily become so too. Hence, calling the discriminants of u and v, Δ, Δ' respectively, it might be inferred that the condition would be of the form

$$\Delta\Theta_2 + \Delta\Delta'\Phi_0 + \Delta'\Theta_1 = 0,$$

where Θ_2 must be of the sixth degree in the coefficients of v and of the second in those of u; Φ_0 of the second degree in those of both, and Θ_1 of the sixth and second degree in those of u and v respectively. The analogy of (18) would suggest the probability that the middle term would be wanting, and that Θ_1, Θ_2 would be the squares of invariants of the third degree in coefficients of one cubic and of the first in those of the other. Consider now the invariants which are the coefficients of the successive powers of λ in the discriminant of $u + \lambda v$,* which may be written

$$(\Delta, 2\Theta, \Phi, 2\Theta', \Delta'\,\overline{\times}\,1, \lambda)^4,$$

* It is evident that we should not look for those invariants of the system of two cubics which are combinants, as likely to enter into Θ_1, Φ_0, Θ_2.

$$\Theta = ad^2a' + a^2dd' + 6(ac^2c' + b^2db') + 4(c^3a' + b^3d') - 3bc(bc' + cb') - 3(abcd' + abdc' + acdb' + bcda'),$$

$$\Theta' = a'd'^2a + a'^2d'd + 6(a'c'^2c + b'^2d'b) + \ldots\ldots - 3(\ldots + b'c'd'a),$$

$$\Phi = a^2d'^2 + d^2a'^2 + \&c.$$

The most probable form of condition that suggests itself for examination is, then, $\Delta\Theta'^2 - \Delta'\Theta^2$.

Now it is evident that if u, v determine an involution, they must be transformable into

$$(A, B, C, D\,|\!|\,xy)^3 \text{ and } (kA, -kB, kC, -kD\,|\!|\,xy)^3$$

by the same linear substitutions. But for these forms it easily appears that $\Delta' = k^4\Delta$, $\Theta' = 8k^3(AC^3 - B^3D) = k^2\Theta$;

so that, eliminating k between these two equations,

$$\Delta\Theta'^2 - \Delta'\Theta^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (22),$$

which is therefore the involution condition of the cubics u, v.

16. This condition may be obtained under a different form by considering the covariants $\overline{1.2}$ and $\overline{1.2^2}$, which are respectively

$$(ab' - a'b,\ 2(ac' - a'c),\ ad' - a'd + 3(bc' - b'c),\ 2(bd' - b'd),\ cd' - c'd\,|\!|\,xy)^4$$

and $$(ac' + a'c - 2bb',\ ad' + a'd - bc' - b'c,\ bd' + b'd - 2cc'\,|\!|\,xy)^2.$$

For the forms of u, v, which express that the roots of one are equal, but of opposite sign, to those of the other respectively, the second and fourth terms of $\overline{1.2}$ vanish, as does the middle term of $\overline{1.2^2}$; so that when u, v determine an involution in which the roots of one correspond to those of the other, each to each, both $\overline{1.2}$ will determine two pairs of corresponding points in the same involution, and $\overline{1.2^2}$ a third pair. Hence the coefficients of $\overline{1.2}$ and $\overline{1.2^2}$ must satisfy the condition (12), *i. e.*, the cubic invariant of $\overline{1.2}$ on these covariants must vanish. Writing, then,

$$l = ab' - a'b,\quad 2m = ac' - a'c,\quad 6n = ad' - a'd + 3(bc' - b'c),$$
$$l' = cd' - c'd,\quad 2m' = bd' - b'd,$$
$$p = ac' + a'c - 2bb',$$
$$2q = ad' + a'd - bc' - b'c,$$
$$p' = bd' + b'd - 2cc',$$

the condition for $\overline{1.2}$ and $\overline{1.2^2}$ on uv determining such an involution as that specified as above—which must plainly be identical with (22)—is obtained by writing l for a, m for b, n for c, m' for d, l' for e, p for a', q for b', and p' for c', in (12); *i. e.*,

$$(l^2m'-3lmn+2m^3)\,p'^3-(l^2l'+2lmm'-9ln^2+6m^2n)\,qp'^2$$
$$+(lml'-3lm'n+2m^2m')\,(4q^2+pp')\,p'-2\,(l'm^2-lm'^2)\,(2q^2+3pp')\,q$$
$$-(lm'l'-3l'mn'+2m'^2m)\,(4q^2+pp')\,p+(ll'^2+2l'mm'-9l'n^2+6m'^2n)\,p^2q$$
$$-(l'^2m-3l'm'n+2m'^3)\,p^3=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (23).$$

This invariant may actually be identified with $\Delta\Theta'^2-\Delta'\Theta^2$ (to a numerical factor) without much labour by taking for one of the cubics (u, suppose) its canonical form ax^3+by^3.

17. The fact of the involution determined by $\overline{1.2}$, $\overline{1.2^2}$ agreeing with that determined by u, v, when the condition (22) or (23) is fulfilled, gives at once the relation between any two corresponding points, whose distances from origin are x', x, and in particular the double points and centre of the involution. For (see § 11) if we form the quadratic covariant $\overline{1.2^2}$ of the covariants $\overline{1.2}$ and $\overline{1.2^2}$ of u, v, it will determine two new corresponding points in the involution. Hence, writing this new covariant, which in full is [see (13)]

$$(lp'+pn-2mq)\,x^2+2\,(mp'+m'p-2nq)\,xy+(np'+l'p-2m'q)\,y^2\ldots\ (24),$$
$$(\mathrm{P},\ \mathrm{Q},\ \mathrm{P}'\nabla xy)^2,$$

the relation between x' and x is (14)

$$2\,(p\mathrm{Q}-\mathrm{P}q)\,x'x+(p\mathrm{P}'-\mathrm{P}p')\,(x'+x)+(\mathrm{P}'q-p'\mathrm{Q})=0;$$

or, substituting from (24) their values for P, Q, P',

$$2(m'p^2+mpp'-3npq-lp'q+2mq^2)x'x+\{l'p^2+2(mp'-m'p)q-lp'^2\}(x'+x)$$
$$-2\,(mp'^2+m'pp'-3np'q-l'pq+2m'q^2)=0 \quad \ldots\ldots\ (25).$$

Making $x'=x$ in (25), and replacing x by $\frac{x}{y}$,

$$(m'p^2+mpp'-3npq-lp'q+2mq^2)\,x^2+\{l'p^2+2(mp'-m'p)-lp'^2\}\,xy$$
$$-(mp'^2+m'pp'-3np'q-l'pq+2m'q^2)\,y^2=0 \quad \ldots\ldots\ (26)$$

determines the foci or double points of the involution; and finally, making x' infinite in (25),

$$2\,(m'p^2+mpp'-3npq-lp'q+2mq^2)\,x+\{l'p^2+2\,(mp'-m'p)-lp'^2\}\,y=0$$
$$\ldots\ldots\ (27)$$

determines the centre.

18. There is one pair of cubics for which the relation (22) or (23) is satisfied, but for which the equations (25), (26), (27) become nugatory. If simultaneously

$$ac'+a'c-2bb'=0,\ \ ad'+a'd-bc'-b'c=0,\ \ bd'+b'd-2cc'=0\ldots.\ (28),$$

the condition (23) is plainly satisfied, since every term contains a vanishing factor. The two cubics therefore form an involution in which the roots of one correspond to those of the other, but the co-

efficients in (25), (26), (27) separately vanish. This case must therefore be dealt with by special considerations.

If the three equations (28) be solved linearly for $\frac{b'}{a'}$, $\frac{c'}{a'}$, $\frac{d'}{a'}$,

$$\frac{b'}{a'} = \frac{abd + b^2c - 2ac^2}{a^2d - 3abc + 2b^3}, \quad \frac{c'}{a'} = \frac{-acd - bc^2 + 2b^2d}{a^2d - 3abc + 2b^3}, \quad \frac{d'}{a'} = \frac{-ad^2 + 3bcd - 2c^3}{a^2d - 3abc + 2b^3}.$$

Hence v is (to a factor) the cubic covariant of u, and each of the points determined by v is harmonic conjugate to one of the three points determined by u with respect to the other two. The Hessian of u is

$$(ac - b^2)\, x^2 + (ad - bc)\, xy + (bd - c^2)\, y^2,$$

and the invariant of this and any quadric $(a_1, b_1, c_1 \mathbin{\text{\textbigcurlyvee}} xy)^2$ is

$$(ac - b^2)\, c_1 - (ad - bc)\, b_1 + (bd - c^2)\, a_1.$$

Again, if u be transformed so that it becomes

$$\mathrm{A}\, x^3 + \mathrm{D}\, y^3,$$

its cubic covariant becomes $\mathrm{AD}\,(\mathrm{A}\, x^3 - \mathrm{D}\, y^3)$,

[which fact verifies that the two cubics determine an involution,] and the invariant above becomes $\mathrm{AD}b_1$;

so that, if $(a_1, b_1, c_1 \mathbin{\text{\textbigcurlyvee}} xy)$ is to determine a pair of corresponding points in the involution determined by u and its cubic covariant, *i. e.*, if the same substitution transforms the quadratic to

$$\mathrm{A}_1 x^2 + \mathrm{C}_1 y^2,$$

the invariant vanishes. Two corresponding points are therefore connected by the relation (between their distances x', x from origin)

$$2\,(ac - b^2)\, x'x + (ad - bc)\,(x' + x) + 2\,(bd - c^2) = 0,$$

and the foci of the involution are determined by

$$(ac - b^2)\, x^2 + (ad - bc)\, xy + (bd - c^2)\, y^2 = 0,$$

i. e., by the Hessian itself of u;* while the centre is given by

$$2\,(ac - b^2)\, x + (ad - bc)\, y = 0.$$

The above formulæ are to be understood as applying to the involution in which the corresponding points are also harmonic conjugates. There are three other involutions determined by the six points; but as these are not analytically distinguishable from one another, the equations giving their foci and centres would be an irresolvable sextic and cubic respectively, into the investigation of which it would occupy too much space here to enter.

* I am not aware that this interpretation of the Hessian of a binary cubic has been previously noticed. Other interpretations have been given by Professors J. H. S. Smith (orally to the London Mathematical Society) and Cayley (Quarterly Journal, Vol. X., p. 148).

19. As an example of a question in cubic curves to which the condition (22) may be applied, suppose it be required

"To find the locus of a point (P) such that, when tangents are drawn from it touching a cissoid in T_1, T_2, T_3, and meeting the same curve again in t_1, t_2, t_3, the pencil joining the cusp (O) with these six points may be in involution, each of the three lines O ($T_1 T_2 T_3$) corresponding to one of the three O ($t_1 t_2 t_3$)."

The Cissoid being $x^3 - y^2 z = 0$, the equation to O (T_1, T_2, T_3) is [Proceedings, Vol. II., p. 162 (8)], $x'y'z'$ being the point P,

$$z'y^3 - 3x'x^2y + 2y'x^3 = 0;$$

that to the three lines O ($t_1 t_2 t_3$) [*ib.* p. 164 (17)]

$$4z'y^3 - 3x'x^2y - y'x^3 = 0.$$

Here $\Delta = 4z'(y'^2z' - x'^3)$, $\Delta' = 16z'(y'^2z' - x'^3)$;

Θ reduces to $ad(ad' + a'd) + 2c^2(3ac' + a'c)$, when $b = 0$, $b' = 0$;

and Θ' „ $a'd'(ad' + a'd) + 2c'^2(ac' + 3a'c)$, „ „ „

so that here $\Theta = 14z'(x'^3 - y'^2z')$, $\Theta' = 2z'(13x'^3 + 14y'^2z')$;

and $\Delta\Theta'^2 - \Delta'\Theta^2 = 432x'^3z'^3(x'^3 - y'^2z')(x'^3 - 28y'^2z')$.

Rejecting the irrelevant factors, the required locus is therefore

$$x^3 - 28y^2z = 0,$$

a curve generated from the cissoid by dividing its ordinates in a constant ratio.

The problem of finding the envelope of a transversal cutting two cubic curves in six points in involution, again, would be solved by substituting in them for z from the equation to the transversal. Applying the condition (22) to the two resulting cubics in x and y, the result would be the tangential equation to the required envelope, which would probably be of the 18th class.

Mr. W. K. Clifford read the following:—

On a Case of Evaporation in the Order of a Resultant.

A particular case of the following theorem was required in the course of my proof that every rational equation has a root; but I have thought that the theorem itself (though indeed a mere obvious remark) was worthy of being placed on record, because of the extremely small number of results of this kind that have yet been arrived at, and of their great importance in analysis.

Theorem. Let it be required to eliminate x between two equations homogeneous in x and certain other variables $y, z, \ldots$, in which equations, however, x only occurs in virtue of the occurrence of a quantity $w = x^\alpha y^\beta z^\gamma \ldots$, where $\alpha + \beta + \gamma + \ldots = \mu$; let also m, n be the orders of

the equations, and h, k the remainders after division of m, n respectively by μ; then the order of the resultant is $= \frac{mn-hk}{\mu}$.

Demonstration. Suppose that p, q are the quotients of the division of m, n respectively by μ; that is to say, let

$$m = p\mu + h, \quad n = q\mu + k,$$

then the two equations may be written

$$a_m + a_{m-\mu} \cdot w + a_{m-2\mu} \cdot w^2 + \ldots + a_h w^p = 0,$$
$$b_n + b_{n-\mu} \cdot w + b_{n-2\mu} \cdot w^2 + \ldots + b_k w^q = 0;$$

where the suffixes of the several coefficients indicate their orders in the variables $y, z, \ldots$ Instead of directly eliminating x from these equations, we may eliminate w; and the result of this may be written down at once by Professor Sylvester's dialytic method. It is in fact

$$\begin{vmatrix} a_m & a_{m-\mu} & \ldots & \ldots & & \\ . & a_m & \ldots & \ldots & & \\ \ldots & \ldots & \ldots & \ldots & & \\ & & & \ldots & \ldots & \\ & & & & b_k & . \\ & & & & b_{k+\mu} & b_k \end{vmatrix}$$

where in the principal diagonal of the determinant the constituent a_m occurs q times, and the constituent b_k occurs p times. The order of the resulting term $(a_m)^q \cdot (b_k)^p$ is $mq + kp$, and this therefore (since the determinant must be homogeneous) is the order of the resultant itself. If we had written the b coefficients before the a coefficients, we should have obtained $np + hq$ as the value of the same quantity. These two values are identical, since by hypothesis

$$q(m-h) = p(n-k) = pq\mu,$$

and therefore $qm + pk = pn + qh = r$, suppose.

Now
$$\begin{aligned} mn &= (p\mu + h)(q\mu + k), \\ &= pq\mu^2 + pk\mu + hq\mu + hk, \\ &= \mu r + hk \end{aligned}$$

$$\therefore \quad r = \frac{mn - hk}{\mu},$$

as was to be proved.

The following extension is brought to light by a different method of proving the original theorem.

Let it be required to eliminate $k-1$ variables $x, y, \ldots$ from k equations homogeneous in these and certain other variables, in which equations, however, $x, y, \ldots$ only occur in virtue of the occurrence of $k-1$ quantities $u, v, \ldots$ all of the same order μ; let also $m_1, m_2, \ldots m_k$ be the orders of the equations, and

$$m_i = p_i \mu + h_i, \quad h_i < \mu;$$

then the order of the resultant is

$$\Pi p\left(\Sigma\frac{h}{p}+\mu\right).$$

For the equations may be written

$$a_h(u,v,\ldots)^p+\theta a_{h+\mu}(u,v,\ldots)^{\ -1}+\ldots+\theta^p a_{h+p\mu}=0,$$

where the h, p are to be affected successively with the suffixes $1, 2, \ldots k$, and θ may be considered $= 1$. Now these equations may be regarded as having coefficients of the constant order h, but the weight of every coefficient of θ^r equal to $r\mu$. This being so, the degree of the resultant in the uneliminated variables will be the sum of its order and weight calculated on these suppositions. But its order is $h_1p_2p_3\ldots p_k$, or $\frac{h_1}{p_1}\Pi p$, due to the coefficients of the first equation, $\frac{h_2}{p_2}\Pi p$ due to the coefficients of the second, and so on; while its weight is $\mu\Pi p$. Hence the entire order of the resultant is

$$\Pi p\left(\Sigma\frac{h}{p}+\mu\right),$$

as stated above.

March 10th, 1870.

Prof. CAYLEY, President, in the Chair.

Mr. E. A. L. Bradshaw Smith was elected a Member, and Messrs. A. and W. M. Ramsay were admitted into the Society. Visitor, Prof. Oppermann of Copenhagen.

Mr. Tucker read two communications by Mr. Clerk-Maxwell: the one on "Topographical Geometry" (on which paper the President and Mr. Archibald Smith made some remarks); the other

On the Displacement in a Case of Fluid Motion.

In most investigations of fluid motion, we consider the velocity at any point of the fluid as defined by its magnitude and direction, as a function of the coordinates of the point and of the time. We are supposed to be able to take a momentary glance at the system at any time, and to observe the velocities; but are not required to be able to keep our eye on a particular molecule during its motion. This method, therefore, properly belongs to the theory of a continuous fluid alike in all its parts, in which we measure the velocity by the volume which passes through unit of area rather than by the distance travelled by a molecule in unit of time. It is also the only method applicable to the case of a fluid, the motions of the individual molecules of which are not expressible as functions of their position, as in the motions due to heat and diffusion. When similar equations occur in the theory of the con-

duction of heat or electricity, we are constrained to use this method, for we cannot even define what is meant by the continued identity of a portion of heat or electricity.

The molecular theory, as it supposes each molecule to preserve its identity, requires for its perfection a determination of the position of each molecule at any assigned time. As it is only in certain cases that our present mathematical resources can effect this, I propose to point out a very simple case, with the results.

Let a cylinder of infinite length and of radius a move with its axis parallel to z, and always passing through the axis of x, with a velocity V, uniform or variable, in the direction of x, through an infinite, homogeneous, incompressible, perfect fluid. Let r be the distance of any point in the fluid from the axis of the cylinder; then it is easy to show that, if x_0 is the value of x for the axis of the cylinder, and x that of the point, and

$$\phi = \frac{a^2}{r^2}(x - x_0), \text{ and } \psi = \left(1 - \frac{a^2}{r^2}\right) y,$$

$\mathrm{V}\phi$ will satisfy the conditions of the velocity-potential, and $\mathrm{V}\psi$ that of the stream function;* and, since the expression for ψ does not contain the time, its value will remain constant for a molecule during the whole of its motion.

If we consider the position of a particle as determined by the values of z, r, and ψ, then z and ψ will remain constant during the motion, and we have only to find r in terms of the time. For this purpose we observe that, if we put ψ in polar coordinates, it becomes

$$\psi = \left(1 - \frac{a^2}{r^2}\right) r \sin\theta,$$

and

$$\frac{dr}{dt} = \frac{\mathrm{V}}{r}\frac{d\psi}{d\theta} = \mathrm{V}\left(1 - \frac{a^2}{r^2}\right)\cos\theta.$$

Expressing $\cos\theta$ in terms of r and ψ, this becomes

$$\frac{dr}{dt} = \frac{\mathrm{V}}{r^2}\sqrt{\{r^4 - (2a^2 + \psi^2)\, r^2 + a^4\}}.$$

If we make $\sqrt{(4a^2 + \psi^2)} + \psi = 2\beta$, and $\frac{a^2}{\beta^2} = c$,

then β will be the value of y when the axis of the cylinder is abreast of the particle, and

$$\frac{dr}{dt} = \frac{\mathrm{V}}{r^2}\sqrt{(r^2 - \beta^2)}\,\sqrt{(r^2 - c^2\beta^2)};$$

* The velocity-potential is a quantity such that its rate of variation along any line is equal to the velocity of the fluid resolved in the same direction. Whenever the motion of the fluid is irrotational, there is a velocity-potential.

The stream function exists in every case of the motion of an incompressible fluid in two dimensions, and is such that the total instantaneous flow across any curve, referred to unit of time, is equal to the difference of the values of the stream function at the extremities of the curve.

and if we now use instead of r a new angular variable χ such that

$$\sin\chi = \frac{\beta}{r} = \frac{\sqrt{(4a^2+\psi^2)}+\psi}{2r},$$

then we can express $\int V dt$ or x_0 in terms of elliptic functions of the first and second kinds,

$$\int V dt = x_0 = \beta\cot\chi\,\sqrt{(1-c^2\sin^2\chi)} + \beta\{E_c(\chi) - F_c(\chi)\},$$

where the position of the axis of the cylinder is expressed in terms of the position of a molecule with respect to it.

Now let us take a molecule originally on the axis of y, at a distance η from the origin, and let the cylinder begin to move from an infinite distance on the negative side of the axis of x; then

$$\psi = \eta, \text{ and } 2\beta = \sqrt{(4a^2+\eta^2)}+\eta, \text{ and } \frac{a^2}{\beta^2} = c;$$

and when the cylinder has passed from negative infinity to positive infinity in the direction of x, then the coordinates of the molecule will be

$$x = \frac{2a}{\sqrt{c}}(F_c - E_c), \quad \text{and} \quad y = \frac{a(1-c)}{\sqrt{c}}.$$

It appears from this expression, that after the passage of the cylinder every particle is at the same distance as at first from the plane of xz, but that it is carried forward in the direction of the motion of the cylinder by a quantity which is infinite when $y = 0$, but finite for all other values of y.

The motion of a particle at any instant is always inclined to the axis of x at double the inclination of the line drawn to the axis of the cylinder. Hence it is in the forward direction till the inclination of this line is 45°, backward from 45° to 135°, and forward again afterwards. The forward motion is slower than the backward motion, but lasts for a longer time, and it appears that the final displacement of every particle is in the forward direction. It follows from this that the condition fulfilled by the fluid at an infinite distance is not that of being contained in a fixed vessel; for in that case there would have been, on the whole, a displacement backwards equal to that of the cylinder forwards. The problem actually solved differs from this only by the application of an infinitely small forward velocity to the infinite mass of fluid such as to generate a finite momentum.

In drawing the accompanying figures, I began by tracing the stream-lines in Fig. 1, p. 86, by means of the intersections of a system of straight lines equi-distant and parallel to the axis, with a system of circles touching the axis at the origin and having their radii as the reciprocals of the natural numbers. (See Prof. Rankine's Papers on Stream-Lines in the "Phil. Trans.")

The cylinder is $\frac{2}{3}$ inch radius, and the stream-lines are originally $\frac{4}{15}$ inch apart.

I then calculated the coordinates, x and y, of the final form of a transverse straight line from the values of the complete elliptic functions for values of c corresponding to every 5°. The result is given in the continuous curve on the left of Fig. 2, p. 87.

I then traced the path of a particle in contact with the cylinder from the equation

$$\tan \tfrac{1}{2}\theta = e^{-\frac{2x_0}{a}},$$

where $x = x_0 + a\cos\theta$ and $y = a\sin\theta$.

The form of the path is the curve nearest the axis in Fig. 3. The dots indicate the positions at equal intervals of time.

The paths of particles not in contact with the cylinder might be calculated from Legendre's tables for incomplete functions, which I have not got.

I have therefore drawn them by eye so as to fulfil the following conditions:—

The radius of curvature is $\frac{1}{2}\frac{a^2y}{a^2\sin^2\theta + y^2}$, which, when y is large compared with a, becomes nearly $\frac{a^2}{2y}$.

The paths of particles at a great distance from the axis are therefore very nearly circles.

To draw the paths of intermediate particles, I observed that their two extremities must lie at the same distance from the axis of x as the asymptote of a certain stream-line, and the middle point of the path at a distance equal to that of the same stream-line when abreast of the cylinder; and, finally, that the distance between the extremities is the same as that given in Fig. 2.

In this way I drew the paths of different particles in Fig. 3. I then transferred these to Fig. 2, to show the paths of a series of particles, originally in a straight line, and finally in the curve already described.

I then laid Fig. 1 on Fig. 2, and drew, through the intersections of the stream-lines and the paths of the corresponding particles in the fluid originally at rest, the lines which show the form taken by a line of particles originally straight as it flows past the cylinder. This method, however, does not give the point where the line crosses the axis of x. I therefore calculated this from the equation

$$x = r + \tfrac{1}{2}a\log\frac{r-a}{r+a},$$

calculating r for values of x differing by $\frac{1}{3}$ inch.

The curves thus drawn appear to be as near the truth as I could get without a much greater amount of labour.

If a maker of "marbled" paper were to rule the surface of his bath with straight lines of paint at right angles, and then to draw a cylindrical ruler through the bath up to the middle, and apply the painted lines to his paper, he would produce the design of Fig. 1, p. 86.

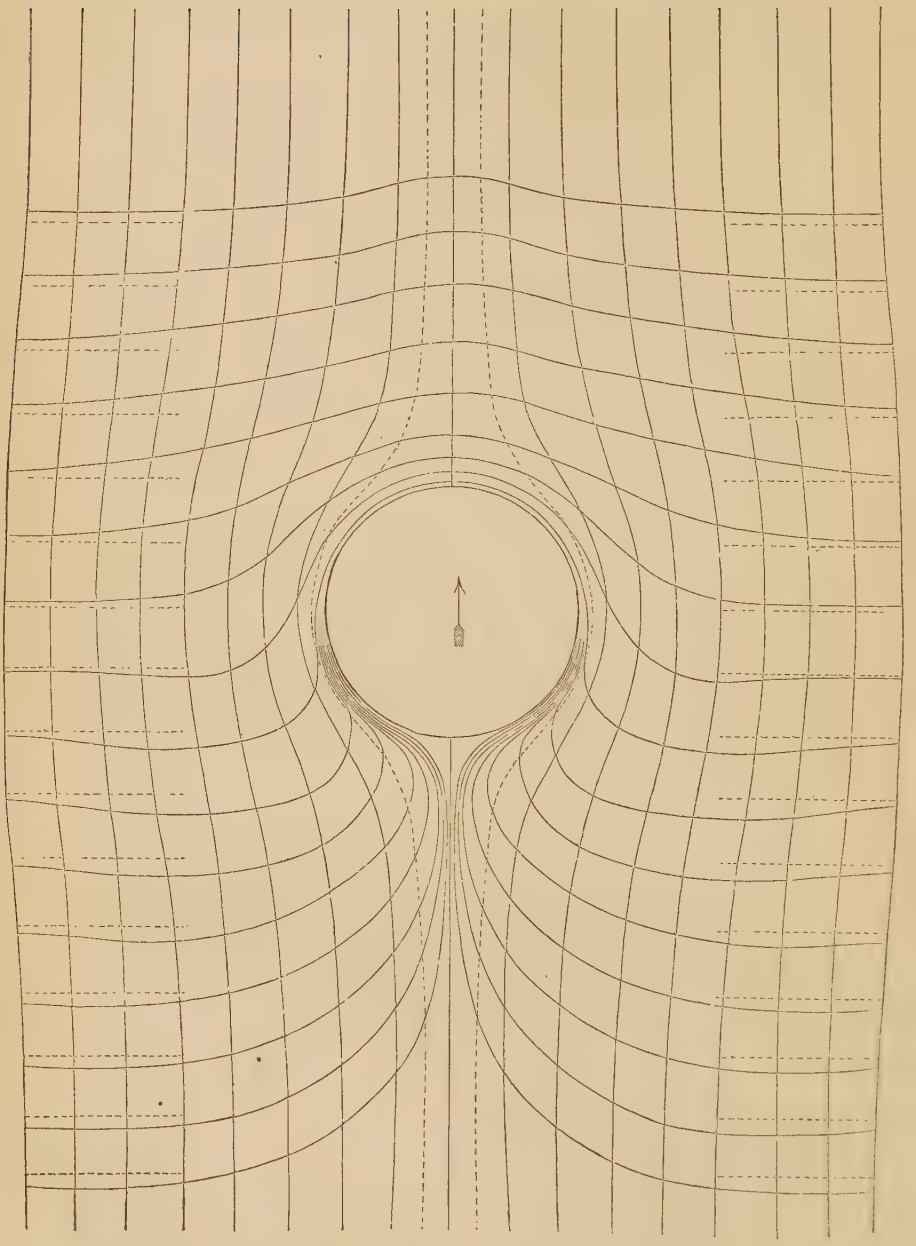


Fig. 1.

Fluid flowing past a fixed cylinder.

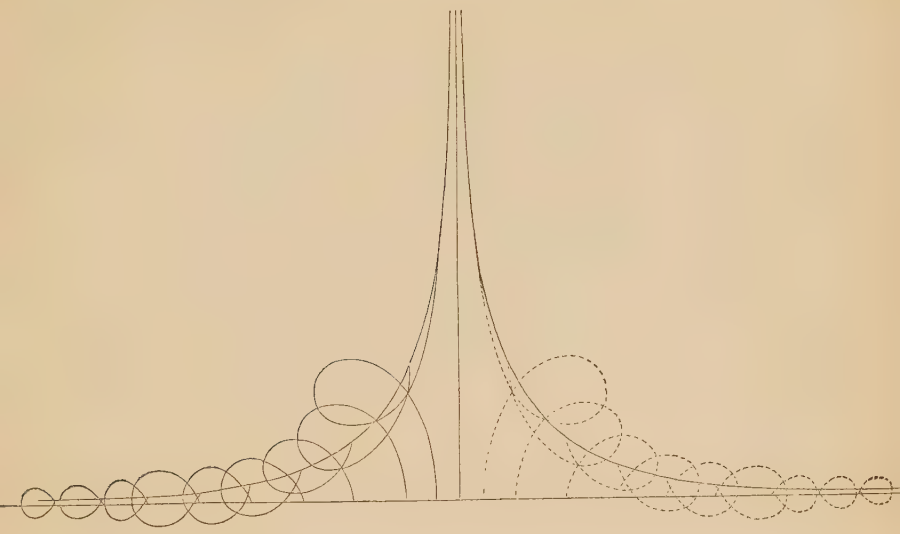


FIG. 2.

Paths of particles of the fluid when a cylinder moves through it.

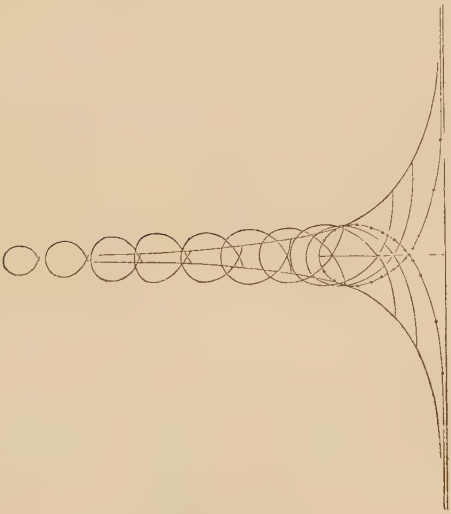


FIG. 3.

Paths of particles at different distances from the cylinder: radius of cylinder, $\frac{2}{3}$ inch. At great distances (β) the path is a circle of radius $\frac{a^2}{2\beta}$, and in this circle $\tan\frac{\theta}{2} = \frac{Vt}{\beta}$.

Mr. S. Roberts then discussed one case in the following paper:—

On the Pedals of Conic Sections.

On a former occasion (Proc. of the Society, Vol. II., p. 135), I have shown, that if two circles are given, one of which passes through the centre of the other, and if a line equal in length to the radius of the latter circle moves with an extremity on each, the locus of any point rigidly connected with the moving line will be composed of a circle and a bicircular quartic having a finite double point.

The equation of the quartic was obtained by dividing out the factor relating to the circle from an equation of the sixth degree. It is better, perhaps, to obtain the quartic equation free from this factor in the following manner, which also presents the result in a more suitable form for discussion:—

1. Draw a circle whose centre is E and radius a, passing through the centre C of another circle whose radius is r. Let AB be a position of the moving line; and let P, rigidly connected with AB, be defined by PD ($=l$), the perpendicular from P on AB, and by the segments AD ($=p$), DB ($=q$), into which AB is divided by the perpendicular.

Denote the vector CP by ρ, and the angles PAB, PBA, ABC, CBE, PCE, which are formed by joining PA, PB, BC, BE, CP, by $\eta, \xi, \chi, \phi, \omega$ respectively.

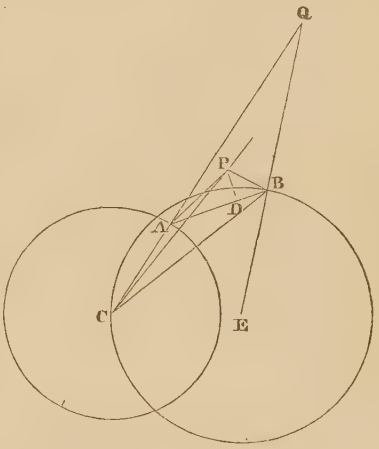


FIG. 1.

Then

$$\rho^2 = \mathrm{PA}^2 + r^2 + 2\mathrm{PA}\, r\cos(2\chi - \eta)$$
$$= l^2 + q^2 + 4rp\cos^2\chi + 4rl\sin\chi\cos\chi.$$

Hence $$(\rho^2 - l^2 - q^2)(1+\tan^2\chi) - 4rl\tan\chi - 4rp = 0 \quad \ldots\ldots\ldots (a).$$

Again $$l^2 + q^2 = \rho^2 + 4a^2\cos^2\phi - 4a\rho\cos\phi\cos(\omega - \phi)$$
$$= \rho^2 + 4a(a - \rho\cos\omega)\cos^2\phi - 4a\rho\sin\omega\sin\phi\cos\phi.$$

Hence

$$(\rho^2 - l^2 - q^2)(1+\tan^2\phi) - 4a\rho\sin\omega\tan\phi + 4a(a - \rho\cos\omega) = 0 \ldots (b).$$

We also have $$a\cos\phi = r\cos\chi \quad \ldots\ldots\ldots\ldots\ldots (c).$$

The equations (a), (b), (c) can be simultaneously satisfied if $\rho^2 = l^2 + q^2$; this is the equation of the circle which we have to exclude, although in a certain way it evidently satisfies the geometrical conditions.

Since, by (c), $\quad a^2(1+\tan^2\chi) = r^2(1+\tan^2\phi)$ ..... (d),
we have, by (a) and (b),

$$r\rho \sin\omega \tan\phi - al\tan\chi - ap - ar + r\rho\cos\omega = 0 \ldots\ldots (e).$$

Again, by comparison of triangles,

$$\rho \sin(\omega-\phi) = \mathrm{PB}\sin(\chi+\xi),$$

or
$$\rho\sin\omega - \rho\cos\omega\tan\phi = \frac{a}{r}(q\tan\chi + l);$$

whence
$$r\rho\cos\omega\tan\phi + aq\tan\chi + al - r\rho\sin\omega = 0 \ldots\ldots (f).$$

Substituting the values of $\tan\chi$, $\tan\phi$ derived from (e) and (f) in (d), we have

$$\left\{\rho^2 - \frac{a}{r}(r+p)\rho\cos\omega - \frac{al}{r}\rho\sin\omega\right\}^2$$

$$-\left\{l\rho\sin\omega - q\rho\cos\omega + \frac{a}{r}(r^2-l^2-p^2)\right\}^2 - \frac{r^2-a^2}{r^2}\{q\rho\sin\omega + l\rho\cos\omega\}^2 = 0$$

..... (A);

or, taking the line of centres for the axis of x, and a perpendicular line through C for the axis of y,

$$\left\{x^2+y^2 - \frac{a}{r}(r+p)x - \frac{al}{r}y\right\}^2 - \left\{ly - qx + \frac{a}{r}(r^2-l^2-p^2)\right\}^2$$

$$-\frac{r^2-a^2}{r^2}\{qy+lx\}^2 = 0 \ldots\ldots (A'),$$

which is the equation of the locus required.

2. We may simplify (A') by transformation. Putting

$$\mathrm{X}' \text{ for } \frac{qx-ly}{\sqrt{(l^2+q^2)}} \quad \text{and} \quad \mathrm{Y} \text{ for } \frac{lx+qy}{\sqrt{(l^2+q^2)}},$$

and afterwards X for $\mathrm{X}' - \frac{a}{r}\frac{(r^2-l^2-p^2)}{\sqrt{(l^2+q^2)}}$,

we obtain

$$\left\{\mathrm{X}^2+\mathrm{Y}^2 + \frac{a}{r}\frac{r^2-l^2-p^2}{\sqrt{(l^2+q^2)}}\mathrm{X} - \frac{2al}{\sqrt{(l^2+q^2)}}\mathrm{Y}\right\}^2$$

$$-(l^2+q^2)\mathrm{X}^2 - \frac{r^2-a^2}{r^2}(l^2+q^2)\mathrm{Y}^2 = 0 \ldots\ldots (\mathrm{B}).$$

The origin is now the real double point, of which the original co-ordinates were

$$x = \frac{aq(r^2-l^2-p^2)}{r(l^2+q^2)},$$

$$y = -\frac{al}{r}\frac{r^2-l^2-p^2}{l^2+q^2}.$$

3. Writing (B) in the form

$$\{\mathrm{X}^2+\mathrm{Y}^2 - \mathrm{AX} - \mathrm{BY}\}^2 - \mathrm{CX}^2 - \mathrm{DY}^2 = 0 \ldots\ldots (\mathrm{B}'),$$

we see that it is the inverse (with respect to a circle whose radius is unity and whose centre is at the origin) of

$$(AX+BY-1)^2-CX^2-DY^2=0,$$

which represents a conic referred to the internal and external bisectors of the angle, real or imaginary, between the tangents through the origin.

When (B) is identified with (B′), we find

$$a^2=\frac{B^2+\{A-\sqrt{(C-D)}\}^2}{4},$$

$$r^2=\frac{C}{C-D}a^2,$$

$$q^2=\frac{C\{A-\sqrt{(C-D)}\}^2}{4a^2},$$

$$l^2=\frac{CB^2}{4a^2}.$$

If the equation of the conic is written in the form

$$(AX+BY-1)^2-(C-D)X^2-D(X^2+Y^2)=0,$$

it appears that a pair of chords joining points of intersection of the conic with the evanescent circle $X^2+Y^2=0$ are given by

$$\{A\pm\sqrt{(C-D)}\}X+BY-1=0.$$

We shall suppose that $\sqrt{(C-D)}$ is real. The conic being real, we must have C or D, or both, positive; and we may assume C positive and not less than D, since the opposite supposition, D positive and not less than C, would only necessitate an interchange of the relations of the axes.

Let P be the perpendicular from the origin on the chord whose equation contains the negative radical, and let θ be the angle which the perpendicular makes with the axis of x. Also let Θ be the angle, real or imaginary, between the tangents to the conic from the origin, the sine of half which angle will be real, but greater than unity, if the angle is imaginary. Then we have

$$a^2=(2P)^{-2},\quad r^2=\sin^2\frac{\Theta}{2}(2P)^{-2},$$

$$q^2=C\cos^2\theta,\quad l^2=C\sin^2\theta;$$

and if X_1, X_2 be the intercepts made by the conic on the axis of X,

$$\frac{1}{X_1}-\frac{1}{X_2}=2\sqrt{C}.$$

We shall find, however, much more direct constructions relative to the reciprocal of the inverted conic.

4. Since we may make use of the other real chord which is given by the equation with a positive radical, there are, in general, two distinct ways of describing the inverse by means of a line moving on two

circles. The other pair of chords of intersections with the evanescent circle $X^2+Y^2=0$ are imaginary. Each pair of chords, however, intersect in a real point, and the pair of points are the intersections of the axes with the polar of the origin, viz., $AX+BY-1=0$. The self-conjugate triangle formed by joining these points and the origin is obviously right-angled at the origin.

5. Since, by the nature of the construction, every real point of the curve described must practically be at a finite distance from the given circles, it follows that we cannot thus trace the inverse of a conic with regard to a point upon itself, which inverse would be a circular cubic curve with a finite double point. The form of the equation of the inverted conic itself shows this, since the absolute term is unity, and the other terms contain no common factor by means of which we can make the absolute term vanish consistently with the conditions.

6. The condition that the conic inverted may be a parabola is

$$4a^2l^2+\frac{r^2-a^2}{r^2}\left\{\frac{a^2}{r^2}(r^2-l^2-p^2)^2-(l^2+q^2)^2\right\}=0;$$

and if this condition is satisfied, the curve described will have a cusp.

A simple case, which can be verified by points, is

$$r=1,\quad a=\sqrt{2},\quad l=2\sqrt{2},\quad q=0;$$

and the associated solution, which gives the same curve, is

$$r=3,\quad a=3\sqrt{2},\quad l=\frac{2\sqrt{2}}{3},\quad q=\frac{8}{3}.$$

The position of the curves will not be the same unless the axes are turned round an angle $\cos^{-1}\frac{1}{3}$.

If we eliminate l, p, q between the expressions in § 2 for the coordinates of the double point and the above condition for a cusp, making use of the identity $p+q=r$, we find that the locus of the cusps is in this case a Limaçon whose equation is $\rho=\frac{2ar}{a^2-r^2}(a-r\cos\theta)$. This, we shall see, is the locus of instantaneous centres. The condition for a cusp may itself be considered as representing a Limaçon relative to the coordinates q and l, its equation being of the form $\rho=\frac{2ar}{a^2-r^2}(r\pm a\cos\theta)$, where ρ is written for l^2+q^2. This curve will also be again encountered.

7. In order that the curve described may be a Limaçon, (*i. e.*, a Cartesian oval with a finite double point,) we must have C=D, or the inverse must be taken from a focus. This implies $r=\infty$, and practically, therefore, this case is excluded; but we are immediately conducted to the common mode of description.

Let A, a point on a given circle ARB, be taken as the centre of the infinite circle. Draw a vector AR, and perpendicular to it the vector AQ. The point Q may be considered as the extremity of the infinite moving line in one of its positions. The direction of the line must be QB, passing through the extremity B of the diameter through A. For the lines AR, QB produced meet at infinity, and form with AQ an isosceles triangle in its limiting condition.

Hence a Limaçon will be described by any point rigidly connected with the line QB, which always passes through the fixed point B, while its extremity Q moves upon the given circle through B.

On making $r=\infty$, equation (B) becomes

$$\left(X^2+Y^2+\frac{2aq}{\sqrt{(l^2+q^2)}}X-\frac{2al}{\sqrt{(l^2+q^2)}}Y\right)^2-(l^2+q^2)(X^2+Y^2)=0.$$

The coordinates of the double point are relative to the original axes

$$x_1=\frac{2aq^2}{l^2+q^2},\quad y_1=-\frac{2alq}{l^2+q^2};$$

therefore

$$x_1^2+y_1^2=\frac{4a^2q^2}{l^2+q^2}=2ax_1.$$

Hence the double point always lies on the given circle.

The above example illustrates the manner in which constructions which involve infinites may be replaced by other finite constructions.

8. If we put $l=0$, $p=-r$, the equation (A) reduces to

$$\rho^2=4r^2\cos^2\omega+4(r^2-a^2)\sin^2\omega,$$

which represents the general central pedal of conic section becoming the Lemniscate of Bernoulli if $a^2=2r^2$. The symmetry of the curve represented shows at once that there are two circles, each passing through a focus and the centre, which may be used in conjunction with a concentric circle whose diameter is the semi-major axis of the conic, for the purpose of describing a central pedal.

9. The general equation of the pedal of a central conic may be written in the form

$$(\rho^2-hx-ky)^2=a_1^2(x-h)^2+b_1^2(y-k)^2.$$

Comparing this with the form

$$\left(X^2+Y^2+\frac{2al}{\sqrt{(l^2+q^2)}}Y-\frac{a}{r}\frac{r^2-l^2-p^2}{\sqrt{(l^2+q^2)}}X\right)^2$$

$$-(l^2+q^2)\left(X-\frac{a}{r}\frac{r^2-l^2-p^2}{\sqrt{(l^2+q^2)}}\right)^2-\frac{r^2-a^2}{r^2}(l^2+q^2)\left(Y+\frac{2al}{\sqrt{(l^2+q^2)}}\right)^2=0,$$

which is (B) transferred to a new origin, we see that the latter locus is the pedal of a central conic, the double point being, of course, the origin of the pedal. The equation of the conic is

$$\frac{X^2}{l^2+q^2} + \frac{\left(Y - \frac{2al}{\sqrt{(l^2+q^2)}}\right)^2}{\frac{r^2-a^2}{r^2}(l^2+q^2)} = 1.$$

From this it follows that

$$l^2+q^2 = (\text{semi-major axis})^2,$$
$$\frac{a^2}{r^2} = (\text{eccentricity})^2.$$

The extraneous circle, included in the general locus, is therefore equal to the circle described upon the major axis as diameter. The conic inverted is the reciprocal of the conic whose pedal is described according to a known general theorem.

9. The next step is to determine, by means of given elements, the construction for drawing the pedal of a conic without tracing the conic itself.

Let O′X, O′Y be the line of the axes of the conic; F a real focus; e the eccentricity, so that $FO' = a_1 e = c$, a_1 being the semi-major axis.

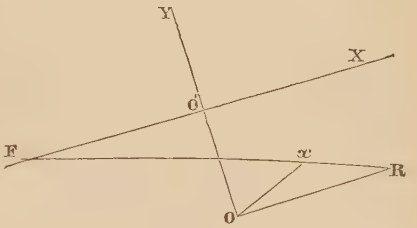


FIG. 2.

Then O, the origin of the (x, y) coordinates in the original investigation of § 1, is on O′Y, the line of the minor axis.* For O is on the line $qx - ly = 0$ or $X = 0$. Also, since the double point or origin of the pedal is on $Y = 0$ relative to O as origin, this point is the foot of the perpendicular let fall from R, the origin of the pedal on OY.

We then have $$a_1^2 = l^2+q^2, \quad \frac{a}{r} = e;$$

and putting L for OR, M for OO′,

$$\frac{2al}{a_1} = M, \quad \frac{2aq}{a_1} - a_1 e = L;$$

therefore $$a = \frac{\sqrt{\{M^2 + (L+c)^2\}}}{2}.$$

Hence the diameter of the circle which passes through the centre of the associated circle in the construction is equal to the distance of the origin of the pedal from a real focus.

* It follows from this, that, for the case of a circular cubic with a finite double point, the centre of the circle whose radius is r passes to infinity, or the circle becomes a line. I have, however, given a method of describing such a cubic curve by finite constructions (Proceedings, Vol. II., p. 129).

Also, since $$\frac{q}{l} = \frac{L+c}{M},$$

the axis of x and FR are equally and oppositely inclined to OR; *i. e.*, if Ox meet FR in x, $Ox = xR$. The construction is now determined.

The coordinates of the centre of the conic are

$$x' = \frac{2al^2}{l^2+q^2}, \quad y' = \frac{2alq}{l^2+q^2}.$$

Hence $x'^2+y'^2 = 2ax'$, and the circle which passes through the centre of the associated circle also passes through the centre of the conic.

10. We have another mode of drawing the same pedal, by joining the origin of the pedal to the other real focus, and constructing as before, but relatively to this new line. There will be a centre common to both pairs of circles, and the ratio of the radii of each pair will be the same, since it represents the eccentricity. This is closely analogous to the double description of an epitrochoid by means of different circles.

11. The method of drawing tangents to a curve described by a point of a moving figure, which has been given by M. Chasles, is immediately applicable to the pedals described in the manner here treated of. We have only to draw the radii to the circles of description at the extremities of the moving line in a given position. The intersection of the radii gives the instantaneous axis of rotation, and a line drawn through it and the corresponding position of the moving point will be the normal, and determines the tangent.

Or if we simultaneously employ the two connected constructions, the intersections of the corresponding radii determine the normal.

12. To find the locus of the instantaneous centres of the moving line AB.

Let CA, EB be produced to meet in Q (Fig. 1); we have to determine the locus of Q. Writing R for the vector CQ, we have

$$R = \frac{2a\cos\phi\sin\phi}{\sin(2\phi-\theta)} = \frac{2a\tan\phi}{\sin\theta(\tan^2\phi-1)+2\cos\theta\tan\phi}.$$

But $$\tan\phi = \frac{a-r\cos\theta}{r\sin\theta}, \text{ since } r\cos(\theta-\phi) = a\cos\phi;$$

$$\therefore \quad R = \frac{2ar\sin\theta(a-r\cos\theta)}{\sin\theta\{(a-r\cos\theta)^2-r^2\sin^2\theta\}+2r\sin\theta\cos\theta(a-r\cos\theta)}$$

$$= \frac{2ar}{a^2-r^2}(a-r\cos\theta).$$

The required locus is therefore a Limaçon having its finite double point at the origin.

Suppose now we refer the locus of instantaneous centres to the line AB as axis. The symmetry of the diagram shows that the locus

relative to the moving figure is also a Limaçon. In fact, B being the pole, its equation is of the form

$$R = \frac{2ar}{a^2 - r^2}(r - a\cos\theta),$$

θ now representing the angle ABE, which is equal to the angle ACE. These loci are, in fact, those noticed in § 6.

13. The motion of a plane in a plane can be generated by the rolling of the curve, which is the locus of instantaneous centres relative to the moving plane, upon the curve which is the locus of instantaneous centres relative to the fixed plane. Consequently the motion generated by the method treated of in this paper can also be generated by the rolling of the Limaçon whose equation is

$$R = \frac{2ar}{a^2 - r^2}(r - a\cos\theta)$$

on the Limaçon whose equation is

$$R = \frac{2ar}{a^2 - r^2}(a - r\cos\theta).$$

We must attend, however, to the relative positions of the curves and to the relative magnitudes of a and r.

If a is $> r$, the fixed curve has a conjugate point and the rolling curve has a node. If r is $> a$, the case is reversed. Suppose that a is $> r$, then CAB can be made linear, in which case B becomes a point of the fixed locus, and the axis AB is inclined to the axis CE at an angle $\cos^{-1}\frac{r}{a}$. Also, since the angle which the tangent at Q makes with CQ is given by

$$\tan\phi = \rho\frac{d\theta}{d\rho} = \frac{a - r\cos\theta}{r\sin\theta},$$

it appears that, in the limiting case, BE is a tangent to the fixed locus, and its angle with AB is $\cos^{-1}\frac{r}{a}$; so that it is also a tangent to the moving locus. Hence the curves are in position for the commencement of the rolling.

14. It is to be observed that, since the trajectories of points of the moving plane are, in the case under consideration, re-entrant, there must be a commensurable relation between the peripheries of the fixed and rolling curves. In fact, the peripheries are equal. For let the equations of two Limaçons be

$$\rho = \pm a + \beta\cos\xi \ldots\ldots (1), \quad \rho = \pm\beta + a\cos\psi \ldots\ldots (2),$$

a being $> \beta$. Then the periphery of (1) is given by

$$2\int_0^{\pi}\sqrt{(a^2 + \beta^2 + 2a\beta\cos\xi)}\,d\xi.$$

And the sum of the peripheries of the loops of (2) is given by

$$2\int_0^{\cos^{-1}\left(\frac{\beta}{a}\right)} \sqrt{(a^2+\beta^2-2a\beta\cos\psi)}\, d\psi$$

$$+2\int_0^{\cos^{-1}\left(-\frac{\beta}{a}\right)} \sqrt{(a^2+\beta^2+2a\beta\cos\psi)}\, d\psi$$

$$=-2\int_\pi^{\cos^{-1}\left(-\frac{\beta}{a}\right)} \sqrt{(a^2+\beta^2+2a\beta\cos\psi')}\, d\psi'$$

$$+2\int_0^{\cos^{-1}\left(-\frac{\beta}{a}\right)} \sqrt{(a^2+\beta^2+2a\beta\cos\psi)}\, d\psi$$

$$= 2\int_0^{\pi} \sqrt{(a^2+\beta^2+2a\beta\cos\psi)}\, d\psi\,;$$

that is to say, the periphery of (1) is equal to the periphery of (2). Hence the trajectories in question repeat themselves after one revolution.

As a particular case, let r be infinite, so that the curves described by points of the moving plane are Limaçons. We have for the fixed curve $\rho = 2a\cos\theta$, and for the moving curve $\rho = -2a$; that is to say, the motion in question will be produced by a circle rolling with internal contact on a circle of half its diameter.

Again, if a be infinite, we have for the fixed curve $\rho = 2r$, and for the rolling curve $\rho = 2r\cos\theta$; hence the motion is produced by a circle rolling internally on a circle of twice its diameter. The hypotrochoids thus produced are concentric and coaxial ellipses, the sum of the lengths of the semi-major axis and the semi-minor axis being constant.

In fact, in this case the pedals degenerate to ellipses, as will be found on substituting a circle and a line through its centre for the auxiliary circles.

When r is $>a$, we may have CEB linear, and a position of the curves at the commencement of rolling may be determined as in the case of $a>r$.

It will be observed that points upon the rolling Limaçon will generate cusped curves (by a known effect of rolling), which result is in accordance with § 6.

15. To find the envelope of the moving line AB.

The line of centres and a line perpendicular to it through C being the axes of x and y, we have for the equation of the line AB,

$$\begin{vmatrix} x, & y, & 1 \\ r\cos\theta, & r\sin\theta, & 1 \\ 2a\cos^2\phi, & 2a\cos\phi\sin\phi, & 1 \end{vmatrix} = 0,$$

or $$r(x\sin\theta - y\cos\theta)(\tan^2\phi+1) + 2a(r\cos\theta - x)\tan\phi + 2a(y - r\sin\theta) = 0,$$

with the equality $$\tan\phi = \frac{a - r\cos\theta}{r\sin\theta}.$$

Substituting and reducing, we have

$$y\cos\theta + \left(\frac{a^2-r^2}{a^2+r^2}x + \frac{2ar^2}{a^2+r^2}\right)\sin\theta - \frac{2a^2r}{a^2+r^2}\cos\theta\sin\theta - \frac{2ar}{a^2+r^2}y = 0.$$

The envelope of a line represented by an equation of this form is (Salmon's "Higher Plane Curves," p. 246)

$$\left[y^2 + \left\{\frac{a^2-r^2}{a^2+r^2}x + \frac{2ar^2}{a^2+r^2}\right\}^2 - \frac{4a^4r^2}{(a^2+r^2)^2} - \frac{16a^2r^2}{3(a^2+r^2)^2}\right]^3$$
$$+27\left[\frac{2ar}{3(a^2+r^2)^2}\left\{y^2 + \left(\frac{a^2-r^2}{a^2+r^2}x + \frac{2ar^2}{a^2+r^2}\right)^2 + \frac{8a^4r^2}{(a^2+r^2)^2} - \frac{32a^2r^2}{(a^2+r^2)^2}\right\}\right.$$
$$\left. - \frac{2a^2r}{a^2+r^2}y\left\{\frac{a^2-r^2}{a^2+r^2}x + \frac{2ar^2}{a^2+r^2}\right\}\right]^2 = 0,$$

which reduces to a point if $r = \infty$, that is, when a point rigidly connected with AB describes a Limaçon.

16. In like manner we may treat the more general case, in which the line whose envelope is required is rigidly attached to the line AB.

The equation of a straight line which divides in a given ratio $m : n$ the line joining two given points whose coordinates are (x_1, y_1), (x_2, y_2), and makes with the same line an angle whose tangent is $\frac{t}{s}$, may be written in the form

$$(m+n)\{(sy+tx)(x_2-x_1) + (ty-sx)(y_2-y_1) + s(x_1y_2 - x_2y_1)\}$$
$$+(m-n)t(y_1y_2 + x_1x_2) - mt(x_2^2 + y_2^2) + nt(x_1^2 + y_1^2) = 0.$$

It is easy to determine the values of $x_2 - x_1$, $y_2 - y_1$, $x_1y_2 - x_2y_1$, &c., in terms of a, r, θ, and we obtain by substitution an equation of the form

$$(m+n)\{(sy+tx)[2ar - (a^2+r^2)\cos\theta] + (ty-sx)(a^2-r^2)\sin\theta$$
$$+2ar\sin\theta(a\cos\theta - r) - 2a^2rt\sin^2\theta\} + ntr(a^2+r^2-2ar\cos\theta) = 0;$$

and the problem is reduced to finding the envelope of a line represented by

$$A\cos 2\theta + B\sin 2\theta + C\cos\theta + D\sin\theta + E = 0.$$

The envelope is still of the 6th degree.

In connection with the subject of this article, the following theorem (due, I believe, to M. Catalan) is important:—

"When a curve rolls on another curve, carrying with it a line, the envelope of this coincides with the envelope of the trajectories of all its points." The theorem has been otherwise stated thus:—"The dif-

ferent points of a moving curve generate a series of curves whose envelope is the same as the envelope of the moving curve." ("Nouvelles Annales," April, 1869.)

17. I may be excused making the rather obvious remark, that the study of the present subject is much facilitated by actually constructing diagrams. These can easily be drawn by points, with the aid of a cardboard triangle as the moving figure. There are details of some interest which are apt to be overlooked in connection with mere analysis, and several particular and limiting cases deserve attention. For instance, making $a=r$, we have the eccentricity $=1$; but the conic is not by this supposition alone a proper parabola. It consists of a line of the length $2\sqrt{(l^2+q^2)}$, representing a conic whose minor axis is evanescent relative to the major axis. The pedal therefore degenerates to two circles, as appears by putting $a=r$ in (A′); and this is in accordance with other known results.

18. The subject may be regarded, in a more general way, as a particular case of the following problem :—Given the paths of two points of an indefinite plane moving in plane space, to find the path of an arbitrary point of the plane.

If $\mathrm{F}(x, y) = 0$, $\phi(x, y) = 0$ are the equations of the given path, we have to eliminate θ from

$$\mathrm{F}(l\cos\theta + p\sin\theta + \mathrm{X},\ \ l\sin\theta - p\cos\theta + \mathrm{Y}) = 0,$$
$$\phi(l\cos\theta - q\sin\theta + \mathrm{X},\ \ l\sin\theta + q\cos\theta + \mathrm{Y}) = 0.$$

Viewed as a particular case of this problem, the general result of the preceding articles is as follows :—

If two points of an indefinite plane, moving in plane space, describe respectively a circle, and one of the circles has its radius equal to the distance between the points and its centre on the other circle; then the other points of the plane describe pedals of conic sections whose minor axes pass through the centre of the first circle, whose centres lie upon the second circle, and whose eccentricities are constant and equal to the ratio of the radius of the second circle to that of the first. Moreover, the origin of each pedal lies upon a line through the centre of the first circle parallel to the major axis of the corresponding conic, and its distance from a real focus is constant and equal to the diameter of the second circle.

For points lying on a circle having its centre at the point whose locus is the second circle, the conics are alike in form and magnitude.

For points lying on the line joining the points which describe the given circles, the double points of the pedals lie upon the line of centres of the circles.

It appears from the nature of the case, that if two of the pedals be

given, the rest will be described by points rigidly attached to a line resting with its extremities on the given pedals; and it is in their capacity of special pedals—namely, pedals of the centre of the first circle considered as an evanescent conic, and of the conic concentric with the first circle and having the diameter of the first circle for its major axis—that the circles become available for the description of pedals.

A discussion arose on the paper, in which the President, Prof. Hirst, Mr. Cotterill, and the Author, took part.

Mr. Archibald Smith then made some remarks "on the Scale for Compensation in the Irish Land Bill."

The following presents were received:—

"Researches respecting Quaternions (First Series), by Sir W. Rowan Hamilton; from the Transactions of the Royal Irish Academy, Vol. XXI., Part II., 1847: presented by R. Hutton, Esq., through Mr. Archibald Smith.

A Portrait of Dr. C. G. J. Jacobi: from Prof. Hirst.

April 14th, 1870.

Prof. H. J. S. SMITH, F.R.S., in the Chair.

The Chairman communicated some remarks bearing on a problem in Kinematics, and Mr. T. Cotterill gave an account of some propositions connected with Residuals and former papers of his own. Mr. Crofton, speaking upon Mr. Cotterill's remarks upon families of curves, all passing through a set of given points, and which it could be proved must also pass through one or more fixed points besides, drew attention to the following theorem:—"If a family of Cartesians have one given focus, and be obliged to pass through three fixed points, then they will all pass through a fourth fixed point, viz., the fourth focus of the circular cubic, which has the three fixed points as foci, and passes through the given focus of the Cartesians." There is also a similar theorem for bicircular quartics. (See No. 11, vol. ii. p. 36.)*

Mr. Jenkins gave a geometrical construction for showing the spherical excess of a triangle, and the Chairman mentioned some focal properties of skew surfaces to which he had been led.

The following presents were received:—

"Monatsbericht," Januar, Februar, 1870; "Crelle," 71 Band, drittes Heft; "Annali di Matematica," tom. iii., Fasc. 3° April, 1870.

* The theorem has been given by Mr. Crofton in the "Educational Times."

May 12th, 1870.

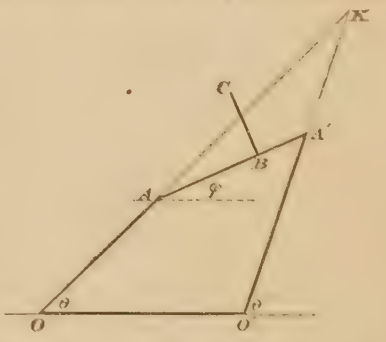
Professor CAYLEY, President, in the Chair.

The Hon. Sir James Cockle, F.R.S., Brisbane, South Australia, was proposed for election.

The President having called upon Mr. Spottiswoode, F.R.S., Vice-President, to take the chair, proceeded to give an account of the following paper :—

On the Mechanical Description of a Nodal Bicircular Quartic.

The ingenious method, devised by Mr. S. Roberts (Proceedings, vol. II. p. 133) for the description of a nodal bicircular quartic suggests a further investigation. We have a quadrilateral OAA′O′, in which the adjacent sides OA, AA′ are equal to each other, and the other two adjacent sides OO′, O′A′ are also equal to each other; O, O′ are fixed points; and we have thus a link AA′, the extremities of which are connected with the radii OA, O′A′ respectively, and consequently describe circles about the centres O, O′ respectively, the radius OA of the one circle being equal to the length AA′ of the link, and the radius O′A′ of the other circle being equal to the distance OO′ of the centres. The theorem is, that any point C, rigidly connected with the link AA′, describes a nodal bicircular quartic, that is, a quartic curve with three nodes (or unicursal quartic), two of the nodes being the circular points at infinity. Any such curve is the inverse of a conic, and it is also the antipode of a conic; viz., if at each point of the curve we draw a line at right angles to the radius vector from the node, these lines envelope a conic having for its pedal the curve in question. It is worth noticing at the outset that to a given position of A′ there correspond two positions of A, viz., the broken line OAA′ may occupy two positions situate symmetrically on the opposite sides of the line OA′. But to a given position of A, there corresponds only one position of A′; viz., the broken line AA′O′ is situate symmetrically with AOO′ on the opposite side of the axis of symmetry O′A; the only other position would be A′ coinciding with O, that is, AA′ with AO, and the locus of C would then be a circle. If the equalities OA = AA′, O′A′ = OO′ did not subsist, then to a given position of A′ there would correspond two positions of A, and to a given



position of A two positions of A′, and the locus of C would be of a higher order than in the actual problem.

I have called AA′ the link; OO′ may be called the bar. OA is then the link-radius, O′A′ the bar-radius; moreover AA′C may be called the constant triangle; and, producing OA, O′A′ to meet in K, then AA′K may be called the variable triangle. Since at any instant the motion of A is normal to KA, and the motion of A′ normal to KA′, it is clear that the motion at that instant of the constant triangle is a motion of rotation about the point K.

Imagine any two positions of the link; say these are A_1A_1' and A_2A_2'. Join A_1A_2, and at its mid-point draw a perpendicular thereto; join in like manner $A_1'A_2'$, and at its mid-point draw a perpendicular thereto; and let these two perpendiculars meet in Γ; we have the two equal triangles $A_1A_1'\Gamma$, $A_2A_2'\Gamma$ (viz., $\Gamma A_1=\Gamma A_2$, $\Gamma A_1'=\Gamma A_2'$, $A_1A_1'=A_2A_2'$) with the common vertex Γ, and which may be brought to coincide with each other by a finite rotation about this point Γ. Considering any particular given position of Γ, if we take the constant triangle AA′C equal to $A_1A_1'\Gamma$ or $A_2A_2'\Gamma$ (viz., $AC=A_1\Gamma$, $A'C=A_1'\Gamma$), then the constant triangle AA′C will, in the course of its motion, come at two different times to coincide with the triangles $A_1A_1'\Gamma$ and $A_2A_2'\Gamma$ respectively; that is, Γ will be a node on the locus described by the point C; and moreover, if K_1 and K_2 be the corresponding positions of K, then by what precedes, the directions of the motion (or tangents at the node) will be normal to $K_1\Gamma$ and $K_2\Gamma$ respectively.

It is to be observed that the point Γ is determined by means of two arbitrary positions A_1A_1', A_2A_2' of the link; that is, the position of Γ depends upon two arbitrary parameters, and therefore Γ may be any point whatever in the plane; if, for an assumed position of Γ, the two positions A_1A_1', A_2A_2' of the link are real, then Γ is a crunode on the locus; but if imaginary, then Γ is an acnode on the locus. The transition case is when the two positions A_1A_1', A_2A_2' coincide with each other, Γ being in this case a cusp on the locus. But from the foregoing general construction for Γ, it appears that when A_1A_1' and A_2A_2' coincide, Γ is in fact the point K, the vertex of the variable triangle. I find that the locus of K is a nodal bicircular quartic, symmetrical in regard to the axis OO′, and having the point O for a node; viz., when, as in the figure, AA′ is $< OO'$, then the point O is an acnode, but when AA′ is $> OO'$, then the point O is a crunode. The curve in question—say the "cuspidal locus"—is a curve such that any point whatever thereof is a cusp on the curve described by some point C; it separates those points Γ, such that each of them is a crunode on the curve described by some point C, from the points Γ which are such that each of them is an acnode of the curve, described by some point C. If (as in the figure) AA′ is $< OO'$, then the cuspidal curve is a closed curve (the inverse of an ellipse), the interior region

being crunodal, and the exterior region acnodal. If AA′ is > OO′, then the cuspidal curve is figure of eight (inverse of a hyperbola), the two interior regions being crunodal, and the exterior region acnodal.

Passing now to the analytical investigation, I take the origin at O, the axis of x being in the direction from O to O′, and the axis of y, at right angles thereto, upwards from O. The inclinations of OA, AA′, O′A′ to the axis Ox, are taken to be θ, ϕ, θ' respectively. I write also $OA=AA'=a$, and $OO'=O'A'=a'$; and

$$m = \frac{a'-a}{a'+a}, \text{ or, what is the same thing,}$$

$$m : 1 : 1+m : 1-m = a'-a : a'+a : 2a' : 2a ;$$

and finally $AB=b$, $BC=c$.

Observing that the angle AA′O′ is $=\theta$, we have $\theta'=\theta+\phi$; and then, in the quadrilateral OAA′O′, the angles A, O′ are $=\pi-\theta+\phi$, $\pi-\theta-\phi$ respectively; whence, projecting on the diagonal OA′, we have

$$a \cos \tfrac{1}{2}(\theta-\phi) = a' \cos \tfrac{1}{2}(\theta+\phi),$$

which, attending to the value of m, is

$$\tan \tfrac{1}{2}\theta \tan \tfrac{1}{2}\phi = m ;$$

whence, writing $\tan \frac{1}{2}\theta = u$, we have $\tan \frac{1}{2}\phi = \frac{m}{u}$,

and the sines and cosines of the angles θ, ϕ, θ' can be all of them expressed in terms of the single parameter u.

For the locus of C we have

$$x = a \cos\theta + b \cos\phi - c \sin\phi,$$
$$y = a \sin\theta + b \sin\phi + c \cos\phi,$$

or, instead of θ, ϕ introducing u, we have

$$\left.\begin{aligned} x &= -a\frac{u^2-1}{u^2+1} + b\frac{u^2-m^2}{u^2+m^2} - c\frac{2mu}{u^2+m^2} \\ y &= \quad a\frac{2u}{u^2+1} + b\frac{2mu}{u^2+m^2} + c\frac{u^2-m^2}{u^2+m^2} \end{aligned}\right\},$$

which, in fact, show that the locus is a bicircular quartic. To put in evidence the third node, I assume that the values belonging thereto are $u=u_1$, $u=u_2$, and that the coordinates of the node are α, β; we have thus

$$\alpha = -a\frac{u_1^2-1}{u_1^2+1} + b\frac{u_1^2-m^2}{u_1^2+m^2} - c\frac{2mu_1}{u_1^2+m^2}$$
$$= -a\frac{u_2^2-1}{u_2^2+1} + b\frac{u_2^2-m^2}{u_2^2+m^2} - c\frac{2mu_2}{u_2^2+m^2},$$

$$\beta = \quad a\frac{2u_1}{u_1^2+1} + b\frac{2mu_1}{u_1^2+m^2} + c\frac{u_1^2-m^2}{u_1^2+m^2}$$
$$= -a\frac{2u_2}{u_2^2+1} + b\frac{2mu_2}{u_2^2+m^2} + c\frac{u_2^2+m^2}{u_2^2+m^2}.$$

These give b, c, α, β in terms of a, m, u_1, u_2; and we may then express the values of $x-\alpha$, $y-\beta$ in terms of a, m, u_1, u_2, u. I find

$$b = \frac{a}{m}\left\{-1+\frac{m+1}{(u_1^2+1)(u_2^2+1)}[(u_1+u_2)^2+(1-m)(1-u_1u_2)]\right\},$$

$$c = \frac{a}{m}\left\{\frac{m+1}{(u_1^2+1)(u_2^2+1)}[-(u_1+u_2)(m-u_1u_2)]\right\};$$

and then

$$x = -a\frac{u^2-1}{u^2+1}$$
$$+\frac{a}{m}\left\{-1+\frac{m+1}{(u_1^2+1)(u_2^2+1)}[(u_1+u_2)^2+(1-m)(1-u_1u_2)]\right\}\frac{u^2-m^2}{u^2+m^2}$$
$$+\frac{a}{m}\left\{\frac{m+1}{(u_1^2+1)(u_2^2+1)}[(u_1+u_2)(m-u_1u_2)]\right\}\frac{2mu}{u^2+m^2},$$

$$y = a\frac{2u}{u^2+1}$$
$$+\frac{a}{m}\left\{-1+\frac{m+1}{(u_1^2+1)(u_2^2+1)}[(u_1+u_2)^2+(1-m)(1-u_1u_2)]\right\}\frac{2mu}{u^2+m^2}$$
$$+\frac{a}{m}\left\{\frac{m+1}{(u_1^2+1)(u_2^2+1)}[-(u_1+u_2)(m-u_1u_2)]\right\}\frac{u^2-m^2}{u^2+m^2},$$

$$\alpha = \frac{a}{m}\frac{m+1}{(u_1^2+1)(u_2^2+1)}(1-u_1u_2)(m+u_1u_2),$$

$$\beta = \frac{a}{m}\frac{m+1}{(u_1^2+1)(u_2^2+1)}(u_1+u_2)(m+u_1u_2);$$

and then

$$(x-\alpha) = -\frac{2(m+1)a}{(u_1^2+1)(u_2^2+1)}\frac{(u-u_1)(u-u_2)}{(u^2+1)(u^2+m^2)}$$
$$\times[(1-u_1u_2)(u^2+m)+(1-m)(u_1+u_2)u],$$

$$(y-\beta) = -\frac{2(m+1)a}{(u_1^2+1)(u_2^2+1)}\frac{(u-u_1)(u-u_2)}{(u^2+1)(u^2+m^2)}$$
$$\times[(u_1+u_2)(u^2+m)-(1-m)(1-u_1u_2)u],$$

where, of course, the factors $(u-u_1)$, $(u-u_2)$ indicate the node (α, β). We have moreover

$$(x-\alpha)^2+(y-\beta)^2 = \frac{4(m+1)^2a^2}{(u_1^2+1)(u_2^2+1)}\frac{(u-u_1)^2(u-u_2)^2}{(u^2+1)(u^2+m^2)},$$

so that, writing

$$\frac{x-\alpha}{(x-\alpha)^2+(y-\beta)^2} = -\frac{1}{2(m+1)a}\frac{[(1-u_1u_2)(u^2+m)+(1-m)(u_1+u_2)u]}{(u-u_1)(u-u_2)},$$

$$\frac{y-\beta}{(x-\alpha)^2+(y-\beta)^2} = -\frac{1}{2(m+1)a}\frac{[(u_1+u_2)(u^2+m)-(1-m)(1-u_1u_2)u]}{(u-u_1)(u-u_2)},$$

we have the locus as the inverse of a conic. To exhibit it as the antipode of a conic, taking X, Y as current coordinates *measured from the node as origin*, the equation of the line through a point of the locus, at right angles to the radius vector from the node, is

$$X(x-\alpha)+Y(y-\beta)-(x-\alpha)^2-(y-\beta)^2=0;$$

or, substituting for $(x-\alpha)$, $(y-\beta)$ their values, this is

$$\begin{aligned}&X[(1-u_1u_2)(u^2+m)+(1-m)(u_1+u_2)u]\\&+Y[(u_1+u_2)(u^2+m)-(1-m)(1-u_1u_2)u]\\&\qquad+2(m+1)a(u-u_1)(u-u_2)=0;\end{aligned}$$

and the antipodal conic is thus the envelope of the line represented by this equation. Putting for shortness

$$\begin{aligned}P&=X(1-u_1u_2)+Y(u_1+u_2),\\Q&=X(u_1+u_2)-Y(1-u_1u_2),\end{aligned}$$

the equation is

$$\begin{aligned}u^2\{P+2(m+1)a\}+u\{(1-m)Q-2(m+1)a(u_1+u_2)\}\\+mP+2(m+1)au_1u_2=0,\end{aligned}$$

and the equation of the conic therefore is

$$\begin{aligned}4\{P+2(m+1)a\}\{mP+2(m+1)au_1u_2\}\\-\{(1-m)Q-2(m+1)a(u_1+u_2)\}^2=0,\end{aligned}$$

so that the conic touches each of the lines $P+2(m+1)a=0$, $mP+2(m+1)au_1u_2=0$ at its intersection with the line $(1-m)Q-2(m+1)au_1=0$. If these lines were constructed, one other condition would suffice for the construction of the conic.

The before-mentioned equations

$$\begin{aligned}\alpha&=\frac{a}{m}\,\frac{m+1}{(u_1^2+1)(u_2^2+1)}(1-u_1u_2)(m+u_1u_2),\\\beta&=\frac{a}{m}\,\frac{m+1}{(u_1^2+1)(u_2^2+1)}(u_1+u_2)(m+u_1u_2),\end{aligned}$$

give

$$\alpha^2+\beta^2=\frac{a^2}{m^2}\,\frac{(m+1)^2}{(u_1^2+1)(u_2^2+1)}(m+u_1u_2)^2;$$

and thence

$$\begin{aligned}\frac{\alpha}{\alpha^2+\beta^2}&=\frac{m}{(m+1)a}\,\frac{1-u_1u_2}{m+u_1u_2},\\\frac{\beta}{\alpha^2+\beta^2}&=\frac{m}{(m+1)a}\,\frac{u_1+u_2}{m+u_1u_2},\end{aligned}$$

which determine u_1+u_2 and u_1u_2 rationally in terms of α, β. For the cuspidal curve, writing $u_1=u_2=v$, we have

$$\begin{aligned}\frac{\alpha}{\alpha^2+\beta^2}&=\frac{m}{(m+1)a}\,\frac{1-v^2}{m+v^2},\\\frac{\beta}{\alpha^2+\beta^2}&=\frac{m}{(m+1)a}\,\frac{2v}{m+v^2},\end{aligned}$$

which show that the cuspidal curve is the inverse of a conic (viz., of an ellipse, if, as in the figure, m is positive). The result in the very same form would be obtained by considering the curve as the locus of the vertex K of the variable triangle.

If we imagine a plane rigidly connected with the link AA′, and carried along with it, then (b, c) are the coordinates of the point C in this moveable plane; and if, as above, (α, β) are the coordinates of the node, then (b, c), and also (α, β), are given functions of (u_1, u_2). We have thus (b, c) functions of (α, β), and reciprocally (α, β) functions of (b, c); that is, we have a correspondence between the points of the fixed plane and those of the variable plane. It is worth while to investigate the nature of this correspondence, although the result does not appear to be one of any elegance.

Writing

$$\frac{(m+1)\,a}{m}\,\frac{\alpha}{\alpha^2+\beta^2} = \mathrm{A},$$

$$\frac{(m+1)\,a}{m}\,\frac{\beta}{\alpha^2+\beta^2} = \mathrm{B},$$

we may, in place of (α, β), consider the point in the fixed plane as given by means of the inverse coordinates (A, B). And then, if $p = u_1+u_2$, $q = 1-u_1u_2$, we have

$$\mathrm{A} = \frac{q}{m+1-q}, \quad \mathrm{B} = \frac{p}{m+1-q};$$

whence

$$p = \frac{(m+1)\,\mathrm{B}}{1+\mathrm{A}}, \quad q = \frac{(m+1)\,\mathrm{A}}{1+\mathrm{A}},$$

$$p^2+q^2 = (1+u_1^2)\,(1+u_2^2) = \frac{(m+1)^2\,(\mathrm{A}^2+\mathrm{B}^2)}{(1+\mathrm{A})^2}.$$

Hence

$$\frac{m}{a}\left(b+\frac{a}{m}\right) = \frac{m+1}{p^2+q^2}\,[p^2-(m-1)\,q],$$

$$-\frac{m}{a}\,c = \frac{m+1}{p^2+q^2}\,p\,(m-1+q),$$

which determine (b, c) in terms of (p, q); that is, of (A, B) or of (α, β).

In reference to some other constructions given in Mr. Roberts' paper, it may be remarked that if we have a moveable plane Π_1 always coincident with a fixed plane Π, and if a condition of the motion is that a circle C_1, fixed in the plane Π_1 and carried along with it, always touches a fixed circle C in the plane Π, then this same condition may be expressed indifferently in either of the forms—(1) a circle C_1 in the plane Π_1 always passes through a fixed point of Π; (2) a point in the plane Π_1 is always situate on a fixed circle C in the plane Π. But if either of the circles C, C_1 reduce itself to a line, then we have two distinct forms of condition; viz., *first*, if a fixed line L_1 in the plane Π_1 always touches a fixed circle C in the plane Π, this is equivalent to the condition that a fixed

line L_1 in the plane Π_1 always passes through a fixed point of the plane Π. And secondly, if a fixed circle C_1 in the plane Π_1 always touches a fixed line L in the plane Π, this is equivalent to the condition that a fixed point in the plane Π_1 is always situate in a fixed line L_1 in the plane Π_1. The different forms of condition therefore are—

(α) A fixed circle C_1 in the plane Π_1 always touches a fixed circle C in the plane Π (where, as above, either circle indifferently may be reduced to a point).

(β) A fixed line L_1 in the plane Π_1 always passes through a fixed point C in the plane Π.

(γ) A fixed point C_1 in the plane Π_1 is always situate in a fixed line L of the plane Π.

Hence, if the motion of the plane Π_1 satisfy any two such conditions (of the same form or of different forms, viz., the conditions may be each α, or they may be α and β, &c.), then the motion of the plane Π_1 will depend on a single variable parameter, and the question arises as to the locus described by a given point or enveloped by a given line of the plane Π; and again of the locus traced out or enveloped on the moving plane Π_1 by a given point of the plane Π. The case considered in the present paper is of course a particular case of the two conditions being each of them of the form (α).

It may be remarked, that if the two conditions be each of them (β), then there will be in the plane Π_1 a fixed point C_1 which describes a circle; and similarly, if the two conditions be each of them (γ), then there will be in the plane Π_1 a fixed point C_1 which describes a circle;* that is, the combination ($\beta\beta$) is a particular case of ($\alpha\beta$), and the combination ($\gamma\gamma$) a particular case of ($\alpha\gamma$).

In a discussion on the paper, Mr. Roberts gave some additional results bearing on the subject, and Mr. Spottiswoode stated that many of the curves drawn (and exhibited) were recognized by him as having come under his notice in the course of experiments he had recently made with elastic strings.

Mr. Roberts then read his Paper

On the Ovals of Descartes.

1. As a contribution to the theory of Cartesian Ovals, I wish (1) to fix the interpretation of the polar equation; (2) to explain a method of description by points, which does not imply the existence of more than one real and finite single axial focus, and is therefore applicable to Cartesians generally, whether the axial single foci are all real, or contain an

* The theorem is, that if an isosceles triangle, on the base AA′ and with angle $= 2\omega$ at the vertex C, slide between two lines OA, OA′ inclined to each other at an angle ω, in such manner that C is the centre of the circle circumscribed about OAA′, then the locus of C is a circle having O for its centre.

imaginary pair; and, by way of illustration, (3) to state some conclusions which afford general interpretations of analytical results incompletely explained or overlooked in consequence of the ambiguity of the polar equation.

Since for a class of Cartesians, viz. those to which I have alluded as having two imaginary axial foci, the vectorial equations so effectively employed by Mr. Crofton (Proc. No. 6, p. 5) become unreal,—that is to say, the coefficients are imaginary in equations representing real curves,—an attempt to interpret the polar equation, which is not subject to this drawback, will not be without utility. I am aware that imaginary vectorial equations, considered as forms derived from real equations, may be interpreted; yet, as remarked by M. Chasles, it is desirable sometimes to use forms which do not imply more than one real axial single focus given. In any case, I may justify myself by another remark of his apropos of these curves,—"On ne saurait avoir trop de moyens differens de décrire une même courbe, parceque chacun exprime une propriété caractéristique de la courbe d'où dérivent naturellement plusieurs autres propriétés qui n'apparaissent pas aussi aisément dans les autres modes de description."

I do not, of course, aim at giving a complete account of the curves in question as far as they are known, and I am afraid that some of the points touched upon are treated with too great conciseness.

2. According to their most general definition, Cartesians are quartic curves having the circular points at infinity for cusps. Their class being 6 and their order 4, they possess only one double tangent. It immediately follows that they consist of not more than two ovals; and, if real, the two ovals lie one within the other. The double tangent also must touch the outer oval only, if the points of contact are real.

Since the circular points at infinity are cusps, the system of Cartesians having the same triple focus* and the same double tangent touching at the same points, real or imaginary, may be represented by $S^2+k(x-l)=0$, where $S=0$ is the equation of a circle, whose centre is at the triple focus and which passes through the points of contact of the double tangent $x-l=0$. Hence, evidently, the curves are symmetrically formed on each side of an axis.

This involves the property, that there are three axial single foci, and that the axis passes through the triple focus. For, generally, if a curve is symmetrical with regard to an axis, and p tangents can be drawn

* In employing the term "triple focus" to designate the intersection of the cuspidal tangents, I follow Dr. Salmon and others. ("Higher Plane Curves," p. 128.) In the case of simply circular curves, the intersections of tangents at the circular points at infinity are analogously called "double foci." In the case of Cartesians, the triple focus is also the centre, and has been so designated. The name "cuspo-focus" has been suggested to me as more satisfactory; this terminology however seems to be incapable of extension.

from a circular point at infinity to the curve, there are p corresponding axial foci; since the two circular points at infinity are symmetrical with regard to any axis.

I have long ago pointed out ("Quarterly Journal of Pure and Applied Mathematics," vol. ii. p. 196) that if any curve has four concyclic foci, there are four sets of concyclic foci lying on four circles which cut one another orthogonally. Consequently, in the present case, since there is in a certain sense a fourth focus on the axis, but infinitely distant,* there are three circles (one of them imaginary) which have their centres on the axis and contain the non-axial single foci of the curve in fours, only 6 of these foci however being finite. Encountering these circles as limiting cases of triconfocal Cartesians, Mr. Crofton has called them confocal circles. It will be convenient to call the three single and (in the case of proper Cartesians) finite axial foci, simply, the axial foci.

3. We may conveniently take the axis of the curve for the axis of x, and a line perpendicular to it through a focus for the axis of y.

Writing ρ^2 for x^2+y^2, we have for the equation of a complete Cartesian, a focus being the pole,

$$[\rho^2-2Bx+C^2]^2-4A^2\rho^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (A).$$

The equivalent form

$$\{(x-B)^2+y^2-B^2-2A^2+C^2\}^2-4A^2(A^2-C^2+2Bx)=0 \ \ldots\ (A')$$

shows that $(x=B,\ y=0)$ are the coordinates of the triple focus, and the line $x=\dfrac{C^2-A^2}{2B}$ is the double tangent. The radius of the circle which passes through the points of contact of the double tangent, and has its centre at the triple focus, is $\sqrt{(B^2+2A^2-C^2)}$. The radius of the corresponding circle, having its centre at the focus pole, is A.

The first polar of the origin is

$$2(\rho^2-2Bx+C^2)(C^2-Bx)-4A^2\rho^2=0.$$

Subtracting this from (A), we get

$$(\rho^2-2Bx+C^2)(\rho^2-C^2)=0.$$

Hence C is the radius of a circle whose centre is at the origin, and which passes through the points of contact of tangents drawn from the focus pole to the curve. The first factor relates to the tangents which make the focus.

The constants of (A) are now defined in relation to the curve.

* As to this, see "Higher Plane Curves," p. 126. Cartesians constitute a sub-class of a class of Bicircular Quartics. In the class, four concyclic foci have become linear. In the sub-class, since only three of these foci are finite, the remaining one is assumed to be infinitely distant. Infinitely distant points are, however, very often analytical conventions. In fact, all points which at infinity remain finitely distant from each other analytically coincide. Thus two parallel straight lines at a distance k from each other can only intersect the line at infinity in points at a distance k from each other; but by a convenient convention, the lines are said to intersect at infinity.

4. We proceed to identify (A) with the vectorial form $m\rho_1-\rho=\kappa$; that is to say, with $m\sqrt{(\rho^2-2kx+k^2)}-\rho=\kappa$,

or $$\left(\rho^2-\frac{2m^2k}{m^2-1}x+\frac{m^2k^2-\kappa^2}{m^2-1}\right)^2-4\frac{\kappa^2}{(m^2-1)^2}\rho^2=0 \quad\ldots\ldots\ldots(B).$$

The equalities $$A^2=\frac{\kappa^2}{(m^2-1)^2},\quad B=\frac{m^2k}{m^2-1},\quad C^2=\frac{m^2k^2-\kappa^2}{m^2-1}$$

give $$k^2-\frac{B^2+C^2-A^2}{B}k+C^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(C),$$

$$\kappa^2+\frac{A^2+C^2-B^2}{A}\kappa+C^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(D),$$

the roots of (C) representing, of course, the two axial foci not at the origin.

There is a noteworthy reciprocity between the forms

$$(\rho^2-2Bx+C^2)^2-4A^2\rho^2=0,$$
$$(\rho^2-2Ax+C^2)^2-4B^2\rho^2=0,$$

which continually reappears.

On the Polar Equation and its interpretation.

5. It is essential to distinguish between the ovals which are said to be conjugate to one another. For this purpose, we may advantageously use the equations

$$\rho^2-2A\rho-2Bx+C^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(E),$$
$$\rho^2+2A\rho-2Bx+C^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(F),$$

the left-hand members of which are the factors of (A). I find it less confusing to keep these forms distinct than to combine them in one sole equation.

If $\rho\cos w$ be written for x, the equations become polar, and the importance of distinctly determining their meaning makes it worth while to dwell on them at some length.

Negative values of ρ indicate that the corresponding lengths are measured on the vector or transversal produced below the axis. Positive values are taken above the axis.

The angle w being taken from 0° to 180° in both equations, we shall have the complete curve. The points determined by (F) are, however, the reflexions, relative to the axis, of the points determined by (E); and, generally speaking, we need only consider specially the latter form.

The portions of the ovals represented by (E) will reach the axis without passing it; they will consist, in fact, of half-ovals.

6. It may be remarked at once, that if C^2 be positive, the signs of the roots of (C) are alike. The origin, therefore, is an *extreme* focus.

On the other hand, if C^2 is negative, the roots of (C) are unlike in sign, and the origin is the *middle* focus.

We will, by convention, measure w from the side of the origin on which the triple focus lies; B will then be essentially positive.

Making $w=0$ and $=\pi$ successively in (E), we have for the axial points of the curve

$$\rho = A+B \pm \sqrt{\{(A+B)^2-C^2\}},$$
$$\rho = A-B \pm \sqrt{\{(A-B)^2-C^2\}}.$$

Consequently, if C^2 is negative, the four intersections are all real. If C^2 is positive, we must have $(A+B)^2 > C^2$ for a real curve. If $A \sim B$ is $>C$, there will be two real ovals; and if $A \sim B$ is $<C$, there will be one real oval only.

The discriminant of (C) is of the form

$$-\frac{(A+B+C)\,(A+B-C)\,(A-B+C)\,(-A+B+C)}{4B^2}.$$

It is easy to see that, $A+B$ being $>C$, the roots of (C) are imaginary if $A \sim B$ is $<C$, and *vice versâ*. Hence, *when there is only one real oval, there are two imaginary axial foci, and vice versâ.*

7. For an angle w, we have

$$\rho = A+B\cos w \pm \sqrt{\{(A+B\cos w)^2-C^2\}}.$$

If there be a passage from real to imaginary values of ρ as the vector moves about the focus pole, such passage will take place when there is a real tangent from the focus pole, and

$$\cos w = \frac{C-A}{B} \text{ or } -\frac{C+A}{B}.$$

There cannot be more than two breaks of this kind. The following schemes show the different cases for C^2 positive, and are obtained by reference to the limiting values of $\cos w$:—

$$A<C$$

$$(1)\quad \begin{cases} \text{No breaks} & C-A>B,\ C+A>B & \ldots\ldots\ldots (a), \\ \text{One break} & C-A>B,\ C+A<B \\ & C-A<B,\ C+A>B & \ldots\ldots\ldots \begin{cases} (b), \\ (c), \end{cases} \\ \text{Two breaks} & C-A<B,\ C+A<B & \ldots\ldots\ldots (d). \end{cases}$$

$$A>C$$

$$(2)\quad \begin{cases} \text{No breaks} & A-C>B,\ C+A>B & \ldots\ldots\ldots (a'), \\ \text{One break} & A-C>B,\ C+A<B \\ & A-C<B,\ C+A>B & \ldots\ldots\ldots \begin{cases} (b'), \\ (c'), \end{cases} \\ \text{Two breaks} & A-C<B,\ C+A<B & \ldots\ldots\ldots (d'). \end{cases}$$

As to (a) and (b), for a real Cartesian, we have seen that $(A+B)^2$ is $>C^2$; this disposes of these cases.

As to (*c*), when A is $<$C, we cannot have A$-$B$>$C and A$>$B, nor can we have B$-$A$>$C, B$>$A, on account of C$+$A$>$B.

As to (*d*), we cannot have A$>$B, nor C$>$B$-$A, B$>$A.

As to (*a'*), we cannot have B$>$A.

As to (*b'*), the conditions are inconsistent.

As to (*c'*), we must have B$>$A, C$>$B$-$A.

And as to (*d'*), we cannot have A$>$B nor C$>$B$-$A.

8. We have now a reduced scheme,

$$(3)\quad \begin{cases} \text{No breaks} & A>B,\ A-B>C, \\ \text{One break} & A\sim B<C, \\ \text{Two breaks} & B>A,\ B-A>C. \end{cases}$$

In the first case, the equation (E) represents two half-ovals on the *same* side of the axis, and therefore the origin must be the *extreme* interior focus (§ 6).

In the second case, we have only one real oval, half of which is represented by (E).

In the third case, the equation (E) represents two half-ovals on *different* sides of the axis, and the origin must be the *external* focus (§ 6).

If C^2 be negative, there are evidently no breaks. The equation (E) represents two half-ovals, on *different* sides of the axis. The origin is the *middle* focus (§ 6).

Let the points in which a vector through the origin meets the part curve represented by (E) be called *corresponding* points. We shall see hereafter (§ 25) that these points are inverse with respect to a definite circle whose centre is the focus pole.

Then, in the first case, corresponding points lie on adjacent parts of the conjugate ovals. In the second case, they lie on the one real oval. In the third case, they lie on one oval. Lastly, when C^2 is negative, they lie on non-adjacent parts of the conjugate ovals.

In what precedes, I have, for simplicity's sake, not specially noticed the case of equalities. These indicate singularities, which do not affect general results. Thus, $C^2=(A-B)^2$ shows that the Cartesian is a Limaçon, and $C^2=(A+B)^2$ indicates a point. If A$=$C, the double tangent passes through the focus pole.

When we have recognized the position of corresponding points for a given focus as pole, the interpretation of the analytical results becomes easy. The conclusions just arrived at are therefore important in many applications.

On the Description of the Curves by the Transformation of a Circle.

9. The equation of a circle

$$\rho^2-2Bx+B^2-R^2=0$$

can be transformed into (E) and (F) by very simple substitutions.

Put $\rho \mp A$ for ρ, and $x-a$ for x. Then we have

$$(\rho \mp A)^2-2Bx+B^2+2Ba-R^2=0,$$

which coincides with (E) and (F) if

$$A^2+B^2+2Ba-R^2=C^2.$$

For brevity, I call a straight line parallel to the axis of y a y-line. Let ρ', x' be the mixed coordinates of a point in (E) and its reflection, and let ρ_1, x_1 be the mixed coordinates of the corresponding point and its reflection on the circle. Then a corresponding point and its reflection on the Cartesian are determined by taking $\rho' = \rho_1 \pm A$, $x' = x+a$; that is to say, by the intersection of the vector $\rho_1 \pm A$ with a y-line separated from the corresponding y-line relative to the circle by a distance a.

We may state the matter thus:—Given a straight line, a circle whose centre is upon the line, and a Cartesian having the line for its axis; we can always determine a point upon the line as pole of the circle, so that to a point of the circle whose coordinates are (ρ, x) shall correspond a point of the Cartesian whose coordinates are $(\rho \pm A, x+a)$ relative to an axial focus as origin.

The distance of the pole of the circle from its centre is evidently equal to the distance of the triple focus from the corresponding focus pole, and A is, as we have seen, the distance of the points of contact of the double tangent from the focus pole, if those points are real.

It is convenient to make corresponding y-lines coincide. This will be the case when the distance between the focus pole and the pole of the circle is a.

The mode of description now given is evidently a modified extension to Cartesian generally of the well known method of describing a Limaçon, the equation of which being written in the form $\rho = \mp A + B \cos w$, suggests the transformation $\rho \pm A$ for ρ in the equation $\rho = B \cos w$. In this case, however, the angles correspond; and in the general case it seems necessary to make the x coordinate correspond.

The auxiliary circle may even degenerate to a point; in which case, however, the construction depends on what is stated in the next article.

10. In the transformation now treated of, a real point on the Cartesian may correspond to an imaginary point on the auxiliary circle. We must take account therefore of the real coordinates (ρ, x) of such a point on the circle. It is evident that a constant length added to ρ may convert an imaginary point into a real one. If it were not easy to find the values of ρ in such cases, this mode of constructing the Cartesian would practically fail. The following methods of finding these values present themselves.

Let O be the pole, C the circle, L the foot of the y-line corresponding to $x = \mathrm{OL}$.

Construction.—Through the pole draw a y-line, and take OM upon it equal to the radius of the circle C. Also through C, the centre of the circle, draw a y-line. Then, with centre L and radius LM, describe a circle meeting the y-line through C in N. CN is the length of the ρ corresponding to $x = \mathrm{OL}$.

For we have, r being the radius of the circle,

$$\mathrm{OL}^2+r^2 = \mathrm{CN}^2+(\mathrm{OL}-\mathrm{OC})^2,$$

or

$$\mathrm{CN}^2-2\mathrm{OC}\,.\,\mathrm{OL}+\mathrm{OC}^2-r^2 = 0.$$

Or we may, with L as centre, describe a circle cutting the given circle C orthogonally. A tangent from the pole to this circle will give the required vector length.

11. I find it best to take $a=\mathrm{A}$, so that we have the forms

$$\rho^2-2\mathrm{A}\rho-2\mathrm{B}x+(\mathrm{A}+\mathrm{B})^2-\mathrm{R}^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(G)},$$

$$\rho^2+2\mathrm{A}\rho-2\mathrm{B}x+(\mathrm{A}+\mathrm{B})^2-\mathrm{R}^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(H)};$$

the pole of the auxiliary circle is here supposed to be intermediate between the focus pole and the centre. The circle is doubly tangential to the curve on the axis, and may be called a diametral circle.

Putting $x=\rho$, we have

$$\rho = \mathrm{A}+\mathrm{B} \pm \mathrm{R},$$

$$\rho = \mathrm{B}-\mathrm{A} \pm \sqrt{\mathrm{R}^2-4\mathrm{AB}}.$$

If R′ be the radius of the diametral circle corresponding to $x = \rho$ in (H), we have

$$\mathrm{R}'^2 = \mathrm{R}^2-4\mathrm{AB}.$$

The condition for imaginary axial foci is therefore

$$\mathrm{R}'^2 = \mathrm{R}^2-4\mathrm{AB} < 0,$$

which, when $(\mathrm{A}+\mathrm{B})^2-\mathrm{C}^2$ is put for R^2, agrees with the condition previously given.

As a matter of construction, it is better to interpret (G) and (H) so that each may represent an oval. To this end, we must include some negative values in (G). For, to complete the outer oval, it will often be necessary to take the difference of the ρ of the circle and A, instead of the sum. This happens, for instance, when the outer oval is indented.

12. If the pole of the auxiliary circle is further from its centre than the focus pole, and on the same side, it is evident geometrically that we have to write $(\mathrm{B}-\mathrm{A})^2-\mathrm{R}^2$ instead of $(\mathrm{A}+\mathrm{B})^2-\mathrm{R}^2$. If the focus pole and the pole of the circle lie on different sides of the centre, we have B negative.

Supposing $\mathrm{R}^2-4\mathrm{AB} > 0$, so that there are two real ovals, there are two diametral circles in each case, by aid of which the Cartesian may be constructed when a focus is given. For if ρ_1, ρ_2 be the axial values

for (G), and ρ', ρ'' those for (H), when $x=\rho$, we may take the diametral circle through (ρ_1, ρ_2), or the diametral circle through (ρ', ρ''), as the auxiliary circle. But it is to be observed, that we cannot for a given focus take at will any one of the six diametral circles as the auxiliary circle.

The matter will be made more clear by a figure (Fig. 1). Let F be the focus pole of the Cartesian, APB the given auxiliary circle whose centre is C. On FC as diameter draw the circle FPC.

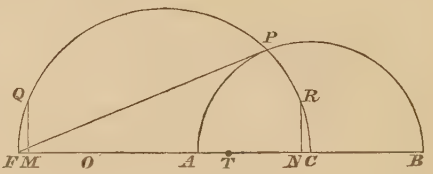


Fig. 1.

Draw the y ordinates of the circle FPC, $QM = RN = \frac{AC}{2}$.

Then if O, the pole of the circle, be anywhere between M, N, we have a Cartesian with a pair of imaginary axial foci. If O be on the left of F, F is the external focus, and the inner oval touches at A, B. If O lie between F and M, there are two ovals, and the circle touches the outer oval. If O be between N and C, there are two ovals, and F is the innermost focus. The circle APB lies between the ovals. If O lie beyond C, the quantity B becomes negative, or the position of the triple focus changes sign relative to FC. This indicates a change of front, so to speak, of the curve. The exterior focus changes sides with respect to the given circle.

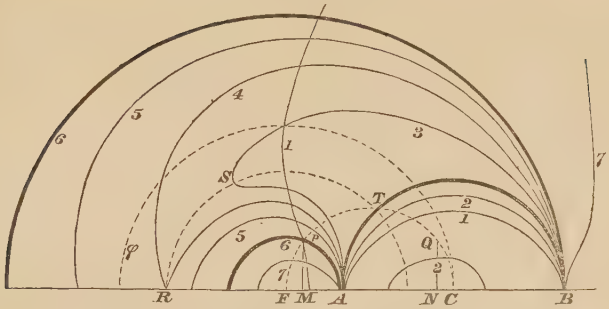
If F is situated within the circle APB, the focus F is the middle one. Taking FT = OC, T is the triple focus. Also FP is the radius C. The circle whose radius is FP also passes through the antifoci of the axial foci not at the origin. This is true also of a circle whose centre is the triple focus and whose radius is FO.

On Certain Systems of Cartesians.

13. In the series of curves which can be thus described, by the change of position of O, we have a focus, a diametral circle, and the axis given. Referring to (E), we have $A \pm B$ given and C^2 given. It follows that the apices relative to the given focus lie on the circle $\rho^2 = C^2$, and the vertices relative to the axis of the antifoci of the axial foci not at the origin are on the circle $\rho^2 = (A \pm B)^2$. The system shows how we pass from a Limaçon having a conjugate point through a series of Cartesians of the second kind to a Limaçon having a node.

The figure (2) will illustrate such a system. The figure is drawn, however, on too small a scale for minute accuracy.

Fig. 2.



The focus pole is F. The auxiliary circle is ATB, whose centre is C. The apices lie on the circle RST. The extra-axial vertices lie on the circle ϕC. The ovals and their conjugates are numbered in the order of the positions of the corresponding pole of the auxiliary circle. Thus, for (1), the pole is to the left of F; for (2), it lies between F and M $\left(\mathrm{MP} = \frac{\mathrm{AC}}{2}\right)$; for (3), between M and N $\left(\mathrm{NQ} = \frac{\mathrm{AC}}{2}\right)$, (3) therefore has no real conjugate oval; for (4), which is a Limaçon, the pole is N; for (5), the pole lies between N and C; and for (6), (concentric circles,) it is at C. The curve (7) shows the effect of taking the pole on the right of C. For the pole M, we should have a Limaçon with a conjugate point, which curve, however, is not drawn.

When two axial foci are imaginary, it cannot be properly said that they are within or without the curve; but it will be observed that their mean, which is real, may lie within or without the oval. The real antifoci are within the oval.

In a similar manner, we can readily construct a diagram to illustrate the system of Cartesians when A and B are given; for in this case we have F, O, C given, and the system of auxiliary circles will be concentric. We shall see that the apices relative to F will lie on a Limaçon, and a set of extra-axial vertices will lie on a circle.

It will be observed that two concentric circles constitute a complete Cartesian. The triple focus and two axial single foci are united at the common centre, and the third axial focus has gone off to infinity.

Fig. 3.

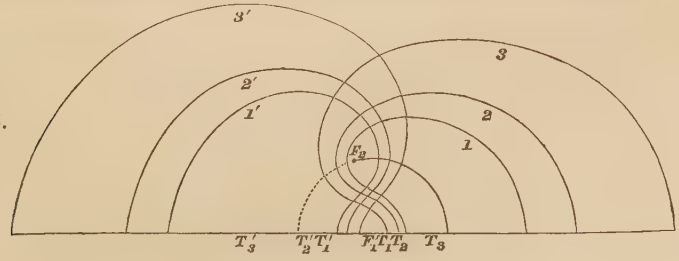


Figure (3) illustrates a system of tri-confocal Cartesians, two of the common axial foci being imaginary. F_1 is the real axial focus, F_2 is one of the real antifoci of the imaginary axial foci. There are then two sub-systems of curves, with respect to one of which systems F_1 lies between the axis of F_2 and the curve, and with respect to the other F_1 lies outside the axis of F_2. These systems cut each other orthogonally, but the curves of one and the same sub-system do not cut each other. The circle described, with F_1 as centre and F_1F_2 as radius, forms a limiting curve, or rather two limiting curves, of the series. In fact, the doubled arcs terminated at the real extra-axial foci constitute degenerate Cartesians. The letters T_1, T_2, T_3, $T_{1'}$, $T_{2'}$, $T_{3'}$ indicate the positions of the triple foci of the curves 1, 2, 3, 1′, 2′, 3′. For the circle, the triple focus unites with F_1. In the case of these Cartesians, the whole series of curves may have real contact with their double tangents. In order that there may be imaginary contact, the angle which F_1F_2 makes with the axis must be smaller than in the figure. For real contact, we must have generally, by (A′),

$$B^2+2A^2-C^2 > \frac{(A^2+2B^2-C^2)^2}{4B^2},$$

or

$$2AB > A^2 \sim C^2.$$

Now, if A is $>$ C, we have, considering the triangle formed by F_1, F_2, and T, $C^2 = A^2+B^2-2AB\cos\theta$, or $A^2-C^2 = 2AB\cos\theta-B^2$. Hence, in this case, the contacts are real. But if C is $>$ A, we have

$$A^2 = C^2+B^2-2BC\cos\phi, \quad \text{or} \quad C^2-A^2 = 2BC\cos\phi-B^2;$$

and for imaginary contacts we must have

$$C\cos\phi > A+\frac{B}{2},$$

which is not the case in the figure. The curves complete themselves symmetrically on the other side of the axis.

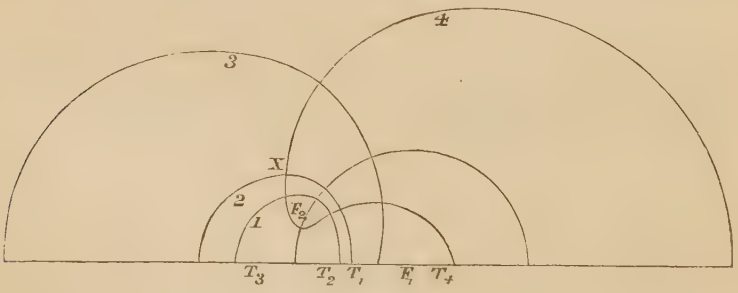


Fig. 4.

Figure (4) relates to a similar triconfocal system, in which, however, curves which have imaginary contact with their double tangents occur.

14. A system of Cartesians represented by $m\rho' - \rho = \kappa$, where ρ', ρ are vectors from two given axial foci, m is constant, and κ variable, may be simply defined as a system having in common two axial foci and the triple focus.

For, comparing (A) and (B), we see that B is constant if m and k are given.

The equation $m\rho' - \rho = 0$ represents a circle having its centre at the triple focus, passing through the antifoci of the given axial foci, and having the latter foci for inverse points. This circle is, in fact, one of the circles connected with the two axial foci, the triple focus, and the focus at infinity (which are four foci on a straight line), in the same manner that the confocal circles are connected with the three finite axial foci and the focus at infinity. In fact, all the circles so depending on the axial foci and the triple focus, taken in fours, stand in a more or less special relation to the Cartesian.

From the equation of the system we have, by differentiation,

$$\frac{m\,(xdx - kdx + ydy)}{\rho'} - \frac{xdx + ydy}{\rho} = 0.$$

The points of contact of tangents from the external focus constitute real apices of the curve. There are also imaginary apices relative to the internal axial foci, and real apices relative to the infinitely distant focus.

Making $\frac{dx}{dy} = 0$, we find that the locus of the points of contact of the double tangents of the system is the circle $m\rho - \rho' = 0$. Also, making $\frac{dy}{dx} = 0$, we find that the locus of the apices relative to the infinitely distant focus is $m^2\,(x-k)\,\rho^2 - x^2\rho'^2 = 0$. This is the locus of the vertex of a triangle, when the base and the ratio of the cosines of the angles at the base are given. The curve is mentioned by Mr. Crofton as the locus of the intersections of pairs of equal tangents to a Cartesian.

This quartic is circular and unicursal, having a double point at infinity. It is, in fact, one of those which I have shown to be traceable by means of a conchoidal motion (Proceedings, Vol. II., p. 125).

If a line RQ moves with its extremity Q on a given line, and is always a tangent to a given circle, the extremity of a line PQ perpendicular to RQ at Q, and equal to half the radius of the circle, will describe the locus in question, P being taken on the same side of RQ as the centre.

The curve consists of two infinite branches which cross at a node, or of two branches which turn back so as not to meet, in which case both the finite double points are conjugate points.

To find the locus of the apices relative to one of the given axial foci, we must eliminate κ between

$$m\rho' - \rho = \kappa, \quad \rho^2 = \frac{m^2k^2 - \kappa^2}{m^2 - 1}.$$

The locus consists of two circles through the focus

$$\rho^2 - kx \pm \frac{k}{m^2 - 1} y = 0.$$

15. The system of Cartesians given by

$$\rho^2 \mp 2A\rho - 2Bx + C^2 = 0,$$

where C^2 is alone variable, may be defined as a system having a common triple focus and a common axial focus, and such that the points of contact of its double tangents lie on a given circle having its centre at the given axial focus.

We get from the general equation, by differentiation,

$$2\left(x\,dx + y\,dy \mp A\frac{x\,dx + y\,dy}{\rho} - B\,dx\right) = 0.$$

Hence putting $\frac{dx}{dy} = 0$, we have for the points of contact of double tangents and the axial points $(\rho \mp A)\, y = 0$;

and for the apices relative to the infinitely distant focus, the locus is

$$(x - B)^2 \rho^2 = A^2 x^2,$$

which represents the conchoid of Nicomedes.

In order to obtain the locus of the apices relative to the common focus, we have to put ρ^2 for C^2 in the equation, and we obtain

$$\rho^2 = Bx \pm A\rho,$$

the well-known equation of a Limaçon of Pascal.

16. The conditions that the two Cartesians

$$\rho^2 \mp 2A\rho - 2Bx + C^2 = 0,$$
$$\rho^2 \mp 2A'\rho - 2B'x + C'^2 = 0,$$

may have the same axial foci are, by (C),

$$C^2 = C'^2, \quad \frac{B^2 + C^2 - A^2}{B} = \frac{B'^2 + C'^2 - A'^2}{B'}.$$

These two conditions express that the apices relative to a focus shall lie on a circle having that focus as centre, and the mean of the two other finite foci shall be the same in all the curves of the system.

On the Normals.

17. The condition that an axial circle may pass through the anti-foci of the axial foci not at the origin, is

$$\frac{C^2 + K^2 - R^2}{K} = \frac{C^2 + B^2 - A^2}{B},$$

the equation of the circle being

$$\rho^2-2Kx+K^2-R^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(K).$$

If (K) intersect (E) in ρ', we have

$$\rho'=\frac{KA\pm BR}{K-B}.$$

For, eliminating x,

$$(K-B)\rho^2-2AK\rho+KC^2-B(K^2-R^2)=0;$$

or, by the condition

$$(K-B)\rho^2-2AK\rho+\frac{K^2A^2-B^2R^2}{K-B}=0.$$

18. Following the analogy of the theory of Conics, we may call the part of the normal line intercepted between the point of the curve through which it is drawn and the axis, the *Normal.**

It is therefore the radius of the axial tangential circle whose centre is its foot. The equation of the circle being

$$\rho^2-2Kx+L^2=0,$$

we have, as we have seen, eliminating x between this and (E),

$$\rho^2-\frac{2AK}{K-B}\rho+\frac{C^2K-BL^2}{K-B}=0.$$

If the circle touches, we have the condition

$$(A^2-C^2)K^2+B(C^2+L^2)K-B^2L^2=0.$$

And to determine ρ when K is given, or *vice versâ*, we have

$$\rho K-B\rho-AK=0.$$

Writing $\rho=K$, we have $K=0$, which relates to the focus, and $K=A+B$; in like manner, writing $\rho=-K$, we have $K=0$ and $K=B-A$. Hence, for the vertices relative to the axis of the antifoci of the axial foci not at the origin, we have the vector and the axial coordinate of the foot of the normal equal. Hence if, with the focus pole as centre, we describe circles whose radii are $A+B$, $B-A$, the chords joining the points where the circles meet the corresponding ovals to the corresponding points where they meet the axis, are normals at the extra-axial vertices. It is here supposed that the focus is external, and therefore B is $>$ A. The other cases can be similarly determined.

We have also

$$\text{Subnormal}=K-x=\frac{\rho^3-3A\rho^2+(C^2+2A^2-2B^2)\rho-AC^2}{2B(\rho-A)}.$$

If N be the length of the normal, we get, writing $L^2=K^2-N^2$,

$$K^3-\frac{C^2+B^2-A^2}{B}K^2+(C^2-N)K+BN=0.$$

* It is evident that real normal distances may correspond analytically to imaginary normal lines.

Making $N=0$, we again obtain the equation (C) which determines the axial foci; and, generally, it appears that *there are three normals of given length, and the mean of the positions of their feet is the mean of the positions of the axial foci.*

If K′, K″, K‴ are the coordinates of the feet of such a set of normals, we have

$$N = -\frac{K'K''K'''}{B}.$$

For points whose vectors are ρ, we have

$$N = \frac{\rho^3 - \frac{C^2+A^2-B^2}{A}\rho^2 + C^2\rho}{\frac{(\rho-A)^2}{A}}.$$

When $N=0$, the equation (D) is satisfied. Hence $-\kappa$ is the value of ρ corresponding to the evanescent normals whose feet are the axial foci not at the origin.

19. The circles which have their centres on the axis are, as is well known, essentially connected in several ways with the Cartesians possessing that axis.

To exemplify this, we may make use of Professor Sylvester's remarkable theorem, that if a Cartesian passes through four concyclic points, these points are concyclic foci of a cubic passing through the foci of the Cartesian. Now since this Cartesian is symmetrical with regard to the axis, it follows that it passes through the four concyclic reflections of the given set of points. Hence the foci are determined as the linear intersections of two circular cubics which are the reflections of each other relative to the axis. These cubics intersect besides in six points on an axial circle. Hence the axis and the system of axial circles constitute the intersections of all such cubics corresponding to Cartesians having that axis.

20. If (E) and (K) intersect in ρ', we have seen (§ 17) that

$$\rho' = \frac{KA \pm BR}{K-B} \quad \ldots\ldots\ldots\

the curve and the other axial foci; and, Tri-confocal Cartesians cut at right angles.

21. If the equation of a Cartesian be written in the form

$$(\rho^2-k^2)^2+A_1x+B_1=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(M),$$

the radius of curvature R is given by means of

$$R^2=\frac{[16y^2(\rho^2-k^2)^2+\{A_1+4x(\rho^2-k^2)\}^2]^3}{16\{(\rho^2-k^2)[16y^2(\rho^2-k^2)^2+\{A_1+4x(\rho^2-k^2)\}^2]+2A_1^2y^2\}^2}.$$

This can be further reduced by means of the equation of the curve.

In this way the square root of the denominator becomes

$$4[\{16x^2(A_1x+B_1)-A_1^2\}(\rho^2-k^2)-8(A_1x+B_1)(A_1x+2B_1)-2A_1^2y^2];$$

and, equated to cipher, this function represents a circular cubic. Hence *the eight points of inflection of a Cartesian lie on a circular cubic.*

To find the radius of curvature at the points of contact of the double tangent.

Putting $\rho^2-k^2=0, \quad y^2=k^2-x^2=\dfrac{A_1^2k^2-B_1^2}{A_1^2},$

we have $$R^2=\frac{A_1^6}{16.4.A_1^4y^4}=\frac{A_1^6}{64(A_1^2k^2-B_1^2)^2}.$$

Comparing the forms (M) and (A′), we find finally

$$R^2=\frac{16B^6A^4}{\{4A^2B^2-(A^2-C^2)^2\}^2}, \quad \text{or} \quad R=\frac{4B^3A^2}{4A^2B^2-(A^2-C^2)^2}.$$

We have $R=\infty$ if $A^2\backsim C^2=2AB$. In this case the points of contact unite at the vertex of the outer oval.

22. If we are given an axial focus, the triple focus, and a point on the curve, as well as the circle having its centre at the axial focus and passing through the points of contact of the double tangent, we have a very simple construction for the normal at the given point. Let fP (Fig. 5) be the vertex of the given point from the given axial focus. Take $PS=A$ the radius of the given circle; then, joining S to the triple focus T, the normal PK is parallel to ST; for we have $\dfrac{fK}{fP}=\dfrac{fT}{fP-PS}$ (§18).

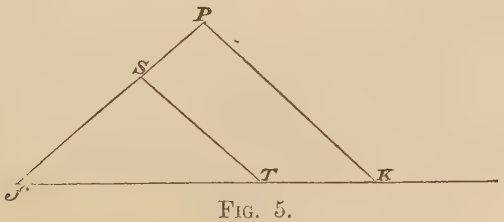


Fig. 5.

23. Denoting by w, w', the angles which the tangents from the external focus make with the axis, we have

$$\cos w = \frac{C-A}{B}, \quad \cos w' = -\frac{C+A}{B}.$$

Hence $$B \sin w \sin w' = \frac{\sqrt{(A^4+B^4+C^4-2A^2B^2-2A^2C^2-2B^2C^2)}}{B}.$$

And we have, by (C),

$$k = B\cos^2\frac{w+w'}{2}, \text{ or } B\cos^2\frac{w-w'}{2}.$$

This gives an easy construction for the remaining foci when the external focus and the triple focus are given. We must draw vectors bisecting the sum and the difference of the angles made with the axis by the tangents to the two ovals from the given focus. Then, projecting the triple focus on these vectors and again on the axis, we have the foci.

Properties deducible from the general interpretation of the Polar Equation.

24. I propose to give some examples by way of illustrating the conclusions of §§ 5—8. It is known that if a transversal meet a complete Cartesian, the sum of the vectors to the points of intersection is algebraically constant. This result must be interpreted differently according to the focus which is taken as pole. Let the pole focus be external, and let the transverse meet both ovals.

To eliminate w between

$$\rho^2-2A\rho-2B\cos w\,\rho+C^2 = 0,$$
$$(a\cos w+b\sin w)\,\rho+c = 0,$$

we have to put the last equation in the form

$$(a\cos w\rho+c)^2-b^2\sin^2 w\,\rho^2 = 0;$$

that is to say, in effect, we take two lines equally inclined to the axis. Since the lines are reflexions of one another relative to the axis, we can refer these intersections to one of them. Remembering this and the nature of corresponding points, we see that, in the case supposed, the sum is $\rho_1+\rho_2-\rho'-\rho''$, the vectors ρ_1, ρ_2 belonging to the intersections on the outer oval, and ρ', ρ'' being the corresponding vectors relative to the inner oval.

But suppose that the transversal meets the same indented oval in four points, then the sum is $\rho_1+\rho_2+\rho_3+\rho_4$. For the equally inclined lines will intersect the oval at points whose vectors are positive.

Again, suppose that the focus pole is the extreme inner one. In this case the vectors are positive; and the sum is $\rho_1+\rho_2+\rho'+\rho''$. And if the pole is the middle focus, we have again the difference

$$\rho_1+\rho_2-\rho'-\rho''.$$

The algebraic sums in question do not therefore comprise all those obtained by alteration of signs.

25. Since the absolute term is the same in (C), (E), and (F), it follows that the product of the vectors of *corresponding points* is equal to the product of the distances of the other two foci from the focus pole.

The interpretation of this for a particular focus as pole, depends immediately upon what has been stated relative to corresponding points.

There are two real circles, and one imaginary circle, having their centres at a focus, and the two other foci for inverse points.

It follows that corresponding points are inverse to each other, with respect to the confocal circle which has its centre at the focus pole.

An oval is therefore inverse to itself relative to the confocal circle whose centre is the external focus. The adjacent parts of conjugate ovals are inverse with respect to the confocal circle whose centre is the extreme inner focus. And the non-adjacent parts of the conjugate ovals are inverse with respect to the imaginary confocal circle.

In the same manner, by reference to corresponding points, we can completely interpret M. Quetelet's theorems, where his statement is adapted to the case of a particular focus as pole.

26. Suppose we have a polar equation of the form

$$\rho^n + (\mathrm{A}\cos w + \mathrm{B}\sin w + \mathrm{C})\,\rho^{n-1}$$
$$+ (a\cos 2w + b\sin 2w + c\cos w + d\sin w + e)\,\rho^{n-2} + \&c. = 0 \ \ldots\ (\mathrm{N}),$$

then the sum of the squares of the roots relative to ρ

$$\Sigma\rho^2 = (\mathrm{A}\cos w + \mathrm{B}\sin w + \mathrm{C})^2 - 2\,(a\cos 2w + b\sin 2w + \&c.)$$

Consequently $\Sigma\int_{w_1}^{w_2} \frac{\rho^2}{2}\,dw$ is integrable in finite terms.

If, therefore, the vector roots represent real points on the branches of the curve, the algebraic sum of the areas intercepted by the different vectors is capable of quadrature when there is no break of continuity.

The origin is a focus, for if odd powers of ρ are present in (N), the complete curve will be represented by a form $\phi^2 - \rho^2\psi = 0$.

The ovals of Cassini, and bicircular quartics generally, present a case in which only even powers of ρ are present.

In the case of a Cartesian represented by (E), we have

$$\frac{\rho'^2 + \rho''^2}{2} = 2\,(\mathrm{A} + \mathrm{B}\cos w)^2 - \mathrm{C}^2$$
$$= (\mathrm{B}^2 + 2\mathrm{A}^2 - \mathrm{C}^2) + 4\mathrm{AB}\cos w + \mathrm{B}^2\cos 2w.$$

Therefore

$$\int \frac{\rho'^2 + \rho''^2}{2}\,dw = (\mathrm{B}^2 + 2\mathrm{A}^2 - \mathrm{C}^2)\,w + 4\mathrm{AB}\sin w + \frac{\mathrm{B}^2}{2}\sin 2w.$$

If the limits are 0, π, the origin being the extreme inner focus, we find that *the sum of the areas of the ovals is equal to twice the area of the circle*

whose centre is at the triple focus, and which passes through the points of contact of the double tangent.

Hence, if we consider the ovals of the locus as forming a ring, a circle, whose radius is determinate, described about the inner oval, will divide the ring into equal portions.

The doctrine of corresponding points on a vector enables us to see that if the origin is external, the areas are intercepted by the same oval; if the origin is the middle focus, the areas are intercepted by non-adjacent parts of the conjugate ovals; if the origin is the innermost focus, they are intercepted by adjacent parts of the ovals.

We may vary the expression of the theorems thence arising in several ways. Thus, let the origin be the external focus, and let two vectors meet the outer oval in (P, Q), (R, S) and the inner oval in (p, q), (r, s), then Area PQpq — Area RSrs can be expressed in finite terms.

27. The equation of a Cartesian being

$$\rho^2 - 2A\rho - 2B\cos w\rho + C^2 = 0,$$

Mr. William Roberts has inferred that the difference of the arcs of the conjugate ovals corresponding to the same position of the radius vector is expressible by an elliptic arc. (Liouville XI., p. 195, quoted by Salmon "Higher Plane Curves," p. 268.)

In fact, the difference of the elements of the arcs is represented by

$$2\sqrt{A^2+B^2-C^2+2AB\cos w}\,.\,dw.$$

Referring again, to what has been said of corresponding points, we are enabled to state the meaning of the analytical result more completely.

For (1), when the focus is external, the corresponding arcs are on the same oval,—Mr. Crofton has remarked this case from a different point of view; (2) if the interior extreme focus is the origin, the corresponding arcs are adjacent portions of the conjugate ovals; (3) if the middle focus is the origin, the corresponding arcs are non-adjacent portions of the conjugate ovals; (4) if two of the axial foci are imaginary, the corresponding arcs lie on the real oval of course, but the difference in question is expressible by two elliptic integrals of the first and second order. For in this case

$$(A \smile B)^2 < C^2; \quad \text{or,} \quad A^2+B^2-C^2 < 2AB.$$

Also, with respect to case (3), it will be observed that it is not really the geometrical difference, but the sum of the corresponding arcs, which is expressed by an elliptic arc. For the corresponding elements are on non-adjacent parts of the conjugate ovals, and one becomes negative to the other.

28. Since, when the foci are real, there are two interior foci, and in

order to obtain the whole length of a semi-oval, we must integrate from $w=0$ to $w=\pi$, we are enabled to draw a remarkable conclusion.

For let the length of the outer oval be 2L, that of the inner $2l$. Then denoting a complete elliptic integral of the second kind by $E_{\frac{1}{2}\pi}$, we have, the extreme inner focus being the origin,

$$L-l = KE_{\frac{1}{2}\pi}.$$

But again, taking the middle forms as origin, we have

$$L+l = K'E'_{\frac{1}{2}\pi}.$$

Hence $$2L = KE_{\frac{1}{2}\pi}+K'E'_{\frac{1}{2}\pi}, \quad 2l = K'E'_{\frac{1}{2}\pi}-KE_{\frac{1}{2}\pi}.$$

That is to say, the lengths of the ovals of a Cartesian are expressed by syzygetic relations between two elliptic quadrants. I confess, however, that a verification of this is desirable.

29. In the case of a nodal Limaçon, the external and middle focus coincide in the node, and we have to modify our integrations. The arc of this curve can be expressed by an elliptic arc in virtue of its character as an Epitrochoid.

Referring the curve to its single focus as origin, we get the equation

$$\rho^2-2A\rho-2B\cos w\rho+(A-B)^2 = 0.$$

Hence the difference between the outer semi-loop and the inner semi-loop is given by

$$2\int_0^{\pi}\sqrt{2AB+2AB\cos w}\,.\,dw = 8\sqrt{AB}\int_0^{\frac{1}{2}\pi}\cos\frac{w}{2}\,.\,d\,.\,\frac{w}{2} = 8\sqrt{AB}.$$

Now, if the equation above given be compared with the form derived from the common equation $\rho = \pm a+b\cos w$, which is transformed into

$$\left(\rho^2-\frac{a^2}{b}x-\frac{b^4-a^4}{4b^2}\right)^2-a^2\left\{\rho^2+\frac{b^2-a^2}{b}x+\frac{(b^2-a^2)^2}{4b^2}\right\} = 0,$$

we see that $B = \dfrac{a^2}{2b}$, $A = \dfrac{b}{2}$, so that we have the theorem,

The difference of the lengths of the loops of a nodal Limaçon is four times the distance between the vertices.

It is remarkable that the difference is independent of the parameter b; if we take the node as pole, the difference in question comes out by means of a formula in the theory of elliptic functions. In consequence of the node, the integrations are not from 0 to π, but from 0 to w_1, and from 0 to w_2, when $\cos w_1 = -\dfrac{a}{b}$, and $w_2+w_1 = \pi$.

We then get a formula

$$L-l = 2(a+b)(2E_{\frac{1}{2}w_1}-E_{\frac{1}{2}\pi}),$$

and the equation of amplitudes for the comparison of elliptic arcs, sometimes written

$$\cos\sigma = \cos\phi\cos\psi-\sin\phi\sin\psi\sqrt{1-e^2\sin^2\sigma},$$

is satisfied. It follows that

$$2E_{\frac{1}{2}w_1} - E_{\frac{1}{2}\pi} = 1 - \sqrt{1-e^2} = \frac{2a}{a+b},$$

and thus the previous result is verified.

30. I may remark that the transformation given in § 9 is a particular case of a more general one. For it is evident that the equation of a Cartesian in the form $\rho^2 - 2A\rho - 2Bx + C^2 = 0$ is transformed into another of similar form by the substitutions $\rho - A'$, $x - a$, for ρ and x respectively. The circle employed in § 9 is itself a Cartesian, and any point may be considered as an axial focus of it, relative to the diameter through the point as axis. In the general case, the pole of the auxiliary equation will be one of the three axial foci. By means of these substitutions, we may derive a Cartesian of the second kind from one of the first kind, and *vice versâ*. It will be observed, however, that a real point on the derived Cartesian may correspond to an imaginary point on the auxiliary Cartesian, as in the particular case when this curve is a circle.

If we are given a parabola, we can also derive Cartesians of both kinds by an obvious and simple transformation.

The equation of the parabola being written in the form

$$y^2 - 2Ay - 2Bx + C^2 = 0,$$

we only have to substitute ρ for y to obtain the equation of a Cartesian. Hence, if the feet of the y ordinates be gathered at the origin, while the extremities slide on perpendiculars to the diameter which is the axis of x, the locus of the extremities will be a Cartesian. The limiting axial points correspond to $y = \pm x$ in the parabola. On account of the identity of the coefficients in the two forms of equation, the theory of Cartesians might be discussed with some advantage by means of the auxiliary parabola.

A slight discussion followed upon the reading of the paper. Dr. Henrici exhibited a plaster cast of the surface

$$xyz - \left(\frac{3}{7}\right)^3 (x+y+z-1)^3 = 0.$$

A cardboard model of this surface was shown to the Society at its meeting in the previous November.

The following presents had been received:—

"Sulle ventisette rette di una Superficie del terzo ordine, nota del Prof. Luigi Cremona;" from the Author.

"Monatsbericht;" März, 1870.

The following Paper, of which an account was given by the Author at the Meeting of the Society on March 11th, 1869, was not received in time to take its place amongst the printed proceedings of that Meeting (Vol. II., p. 174) :—

ON THE RATIONAL TRANSFORMATION BETWEEN TWO SPACES.

By Prof. CAYLEY.

Two figures are rationally transformable each into the other (or, say, there is a rational transformation between the two figures) when to a variable point of each of them there corresponds a single variable point of the other. The figures may be either loci in a space, or *locus in quo* of any number of dimensions; or they may be such spaces themselves. Thus the figures may be each a line (or space of one dimension), each a plane (or space of two dimensions), or each a space of three dimensions: these last are the cases intended to be considered in the present Memoir, which is accordingly entitled, "On the Rational Transformation between Two Spaces." I observe in explanation (to fix the ideas, attending to the case of two planes), that any rational transformation between two planes gives rise to a rational transformation between curves in these planes respectively (one of these curves being any curve whatever): but *non constat*, and it is not in fact the case, that every rational transformation between two plane curves thus arises out of a rational transformation between two planes. The problem of the rational transformation between two planes (or generally between two spaces) is thus a distinct problem from that of the rational transformation between two plane curves (or loci in the two spaces respectively).

I consider,—in the Memoir, (1) the rational transformation between two lines; this is simply the homographic transformation: (2) the rational transformation between two planes; and here there is little added to what has been done by Prof. Cremona in his memoirs, "Sulle Trasformazioni Geometriche delle Figure Piane," (Mem. di Bologna t. II., 1863, and t. V., 1865; see also "On the Geometrical Transformation of Plane Curves," British Assoc. Report, 1864): (3) the rational transformation between two spaces; in regard hereto I examine the general theory, but attend mainly to what I call the lineo-linear transformation; viz., it is assumed that the coordinates of a point in the one space, and the coordinates of the corresponding point in the other space are connected by three lineo-linear equations (that is, each equation is linear in the two sets of coordinates respectively). The lineo-linear transformation presents itself in the preceding two cases; viz., between two lines, the homographic transformation (which, as already mentioned, is the only rational transformation) is lineo-linear;

and between two planes, the lineo-linear transformation is in fact the well known inverse transformation $\left(x' : y' : z' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$. As regards two spaces, the lineo-linear transformation has not, I think, been discussed in a general manner, and it gives rise to a theory of some complexity, and of great interest.

The General Principle of the Rational Transformation between Two Spaces.

1. In all that follows, the two spaces (lines, planes, or three dimensional spaces, as the case may be), or any corresponding loci in the two spaces respectively, are referred to as the first and second figures respectively. The two figures are in general considered, not as superimposed or situate in a common space, but as existing, each independently of the other, as a separate *locus in quo* or figure in such locus. The unaccented coordinates (x, y), (x, y, z), or (x, y, z, w), as the case may be, refer throughout to a point of the first figure; the accented coordinates refer in like manner to the corresponding point of the second figure.* Moreover X, Y, ... are used to denote functions of the same order, say n, of the coordinates $(x, y, \ldots)$: viz., (X, Y) are each of them of the form $(* \mathfrak{f}(x, y)^n$; (X, Y, Z) each of the form $(* \mathfrak{f}(x, y, z)^n$, (X, Y, Z, W) each of the form $(* \mathfrak{f}(x, y, z, w)^n$, as the case may be; and in like manner X′, Y′, ... are used to denote functions of the same order, say n', of the coordinates $(x', y', \ldots)$. This being so—

The condition of a rational transformation is that we have simultaneously $x' : y' \ldots = X : Y \ldots$; $x : y \ldots = X' : Y' \ldots$ viz., these equations must be such that either set shall imply the other set.

2. If, to fix the ideas, we attend to the case of two planes, or take the sets to be $x' : y' : z' = X : Y : Z$; $x : y : z = X' : Y' : Z'$,

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* The coordinates (x, y) of a point in a line may be conceived as proportional to given multiples (α times, β times) of the distances of the point from two fixed points on the line; similarly the coordinates (x, y, z) of a point in a plane as proportional to given multiples (α times, β times, γ times) of the perpendicular distances of the point from three fixed lines in the plane; and the coordinates (x, y, z, w) of a point in a space as proportional to given multiples (α times, β times, γ times, δ times) of the perpendicular distances of the point from four fixed planes in the space. Observe that even if the coordinates (x, y) and (x', y') refer to the same line, and to the same two fixed points in this line, they are not of necessity the same coordinates: viz., the factors for x, y may be α, β, and those for x', y' may be α', β'. If these are proportional (viz., if $\alpha : \beta = \alpha' : \beta'$), then (x', y') will be the same coordinates of P′ that (x, y) are of P; and in this case, but not otherwise, the equation $xy' - x'y = 0$ will imply the coincidence of the points P, P′. The like remarks apply to the coordinates (x, y, z) and (x, y, z, w).

then starting with the set $x':y':z' = X:Y:Z$, for any given point (x, y, z) whatever in the first figure, we have a single corresponding point (x', y', z') in the second figure; but for any given point (x', y', z') in the second figure, we have *primâ facie* a system of n^2 points in the second figure, viz., these are the common points of intersection of the curves $x':y':z' = X:Y:Z$ (in which equations x', y', z' are regarded as given parameters, x, y, z as current coordinates, and the equations therefore represent curves of the order n in the first figure). The curves may however have only a single variable point of intersection; viz., this will be the case if each of the curves passes through the same n^2-1 fixed points (points, that is, the positions of which are independent of x', y', z'); and in order that the curves in question may each pass through the n^2-1 points, it is necessary and sufficient that these shall be common points of intersection of the curves $X=0$, $Y=0$, $Z=0$. [Observe that the condition thus imposed upon the curves $X=0$, $Y=0$, $Z=0$ will in certain cases imply that the curves have n^2 common intersections; or what is the same thing, that the functions X, Y, Z are connected by an identical equation, or syzygy, $\alpha X+\beta Y+\gamma Z = 0$. This must not happen; for if it did, not only there will be no variable point of intersection, and the transformation will on this account fail; but there would also arise a relation $\alpha x'+\beta y'+\gamma z' = 0$ between (x', y', z'), contrary to the hypothesis that (x', y', z') are the coordinates of any point whatever of the second figure. It thus becomes necessary to show that there exist curves $X=0$, $Y=0$, $Z=0$, satisfying the required condition of the n^2-1 common intersections, but without a remaining common intersection, or, what is the same thing, without any syzygy $\alpha X+\beta Y+\gamma Z = 0$.]

3. The curves $x' : y' : z' = X : Y : Z$ having then a single variable point of intersection, if we take (x, y, z) to be the coordinates of this point, the ratios $x : y : z$ will be determined rationally; that is, as a consequence of the first set of equations, we obtain a second set $x : y : z = X' : Y' : Z'$, where X', Y', Z' will be rational and integral functions of the same order, say n', of the coordinates (x', y', z'); that is, we have a second set of equations, and consequently a rational transformation, as mentioned above.

4. It is easy to see that we have $n'=n$; in fact, consider in the first figure a curve $\alpha X+\beta Y+\gamma Z = 0$, and an arbitrary line $ax+by+cz = 0$; to these respectively correspond, in the second figure, the line $\alpha x'+\beta y'+\gamma z' = 0$, and the curve $aX'+bY'+cZ' = 0$; the curves are of the orders n, n' respectively, or the curve and line of the first figure intersect in n points, and the line and curve of the second figure intersect in n' points; which two systems of points must correspond point to point to each other; that is, we must have $n'=n$. It will

presently appear how different the analogous relation is in the transformation between two spaces.

5. Ascending to the case of two spaces, we have here the two sets

$$x' : y' : z' : w' = X : Y : Z : W; \quad x : y : z : w = X' : Y' : Z' : W',$$

the theory is analogous; the surfaces $x' : y' : z' : w' = X : Y : Z : W$ (surfaces of the order n in the first figure) must have a single variable point of intersection, and they must therefore have a common fixed intersection equivalent to n^3-1 points of intersection: I say *equivalent* to n^3-1 points, for this fixed intersection need not be n^3-1 points, but it may be or include a curve of intersection.* The surfaces $X=0$, $Y=0$, $Z=0$, $W=0$ must consequently have a common intersection equivalent to n^3-1 points; there is (as in the preceding case) a cause of failure to be guarded against, viz., the condition as to the intersection must not be such as to imply one more point of intersection, that is, to imply an identical equation or syzygy $\alpha X+\beta Y+\gamma Z+\delta W=0$ between the functions X, Y, Z, W; but it is assumed that they are not thus connected. There is then a single variable point of intersection of the surfaces $x' : y' : z' : w' = X : Y : Z : W$; or taking the coordinates of this point to be (x, y, z, w), we have the ratios $x : y : z : w$ rationally determined; that is, we have a second set of equations $x : y : z : w = X' : Y' : Z' : W'$, where X', Y', Z', W' are rational and integral functions of the same order, say n', in the coordinates (x', y', z', w'); viz., we have the rational transformation, as above, between the two spaces.

6. Suppose that the common intersection of the surfaces $X=0$, $Y=0$, $Z=0$, $W=0$ is or includes a curve of the order ν; and consider in the first figure the two surfaces

$$\alpha X+\beta Y+\gamma Z+\delta W=0, \quad \alpha_1 X+\beta_1 Y+\gamma_1 Z+\delta_1 W=0,$$

and the arbitrary plane $ax+by+cz+dw=0$. The two surfaces intersect in the fixed curve ν, and in a residual curve of the order $n^2-\nu$; hence the two surfaces and the plane meet in ν points on the fixed curve, and in $n^2-\nu$ other points. Corresponding to the surfaces and plane in the first figure, we have in the second figure the two planes $\alpha x'+\beta y'+\gamma z'+\delta w'=0$, $\alpha_1 x'+\beta_1 y'+\gamma_1 z'+\delta_1 w'=0$, and the surface $aX'+bY'+cZ'+dW'=0$ of the order n': these intersect in n' points, being a system corresponding point to point with the $n^2-\nu$ points of the first figure; that is, we must have $n'=n^2-\nu$. And conversely, it follows that in the second figure the common intersection of the surfaces $X'=0$, $Y'=0$, $Z'=0$, $W'=0$ will be or include a

* The curve of intersection may consist of distinct curves; each or any of which may be a singular curve of any kind in regard to the several surfaces.

curve of the order ν'; and that we shall have $n=n'^2-\nu$ Hence also

$$\nu-\nu' = (n-n')\ (n+n'+1).$$

7. The principle of the rational transformation comes out more clearly in the foregoing two cases than in the case of two lines, which from its very simplicity fails to exhibit the principle so well; and I have accordingly postponed the consideration of it: but the theory is similar to that of the foregoing cases. We must have the two sets (each a single equation) $x' : y' = X : Y$, and $x : y = X' : Y'$. The equation $x' : y' = X : Y$ must give for the ratio $x : y$ a single variable value; viz., there must be $n-1$ constant values (values, that is, independent of x', y'); this can only be the case by reason of the functions having a common factor M of the order $(n-1)$; but this being so, the common factor divides out, and the equation assumes the form $x' : y' = X : Y$, where X, Y are linear functions of (x, y): and we have then reciprocally $x : y = X' : Y'$, where X', Y' are linear functions of (x', y'). Thus in the present case, instead of an infinity of transformations for different values of n, n', we have only the well known homographic transformation wherein $n=n'=1$.

8. In the discussion of the foregoing cases of the transformation between two planes and two spaces, it was tacitly assumed that n was greater than 1, and the transformations considered were thus different from the homographic transformation; but it is hardly necessary to remark that the homographic transformation applies to these cases also; viz., for two planes we may have $x' : y' : z' = X : Y : Z$, and $x : y : z = X' : Y' : Z'$, where (X, Y, Z), (X', Y', Z') are linear functions of the two sets of coordinates respectively; and similarly for two spaces $x' : y' : z' : w' = X : Y : Z : W$ and $x : y : z : w = X' : Y' : Z' : W'$, where (X, Y, Z, W), (X', Y', Z', W') are linear functions of the two sets of coordinates respectively. We may, if we please, separate off the homographic transformation (as between two lines, planes, and spaces respectively), and restrict the notion of the rational transformation to the higher or non-linear transformations; in this point of view, the case of two lines would not be considered at all, but the theory of the rational transformation would begin with the case of the two planes. Such severance of the theory is, however, somewhat arbitrary; and moreover the homographic transformation between two lines (being, as mentioned, the only rational transformation) is analogous not only to the homographic transformation between two planes, and to the homographic transformation between two spaces, but it is also analogous to the lineo-linear (or quadric) transformation between two planes, and to the lineo-linear (which is a cubic) transformation between two spaces.

9. For the sake of bringing out this analogy, I shall consider in some detail the homographic transformation between two lines: but as regards the homographic transformations between two planes and between two spaces respectively (although there is room for a like discussion) the theories may be considered as substantially known, and I do not propose to go into them.

The Homographic Transformation between Two Lines.

10. By what precedes, it appears that we have $x' : y' = X : Y$, where (X, Y) are linear functions of (x, y); and conversely, $x : y = X' : Y'$, where X', Y' are linear functions of (x', y'); or what is the same thing, the relation is expressed by a single equation

$$(ax+by)\,x' + (cx+dy)\,y' = 0;$$

or, as this may conveniently be written,

$$\left(\begin{smallmatrix} a, & b \\ c, & d \end{smallmatrix}\right)(x, y)(x', y') = 0,$$

or, when the expression of the actual values of the coefficients is unnecessary,

$$(*\,(x, y)(x', y') = 0.$$

We thus see that the rational transformation between two lines is in fact the homographic transformation; and also that it is the linear-linear transformation.

11. A special case is when $ad-bc = 0$.

Writing here

$$\frac{c}{a} = \frac{d}{b} = m, = \frac{b'}{a'},$$

the equation is $(ax+by)\,(x'+my') = 0,$

that is $(ax+by)\,(a'x'+b'y') = 0;$

viz., either $ax+by = 0$, without any relation between x', y'; or else $a'x'+b'y' = 0$, without any relation between x, y: that is, to the single point $ax+by = 0$ of the first figure there corresponds any point whatever of the second figure; and to the single point $a'x'+b'y' = 0$ of the second figure there corresponds any point whatever of the first figure.

12. In the general case where $ad-bc \neq 0$, we may either by a linear transformation ($ax+by$, $cx+dy$ into $y, -x$ or into $x, -y$) of the coordinates of a point of the first figure, or by a linear transformation ($a'x'+c'y'$, $b'x'+d'y'$ into $y', -x'$ or into $x', -y'$) of the coordinates of a point in the second figure (or in a variety of ways by simultaneous linear transformations of the two sets of coordinates) transform the relation indifferently into either of the forms $xy' - x'y = 0$, $xx' - yy' = 0$; the former of these, or $x : y = x' : y'$, is the most simple expression of the homographic transformation; the latter, or $x' : y' = \frac{1}{x} : \frac{1}{y}$, is its expression as an inverse transformation.

13. If, to fix the precise signification of the coordinates (x, y), we employ the distances from a fixed point O in the line; taking the distances of the two fixed points (say A, B) to be α, β, and that of the variable point P to be ρ, then we have x, y proportional to given multiples $p(\rho-\alpha)$, $q(\rho-\beta)$ of the distances from the two fixed points; or writing $\frac{p}{q} = n$, we may say that the coordinate $\frac{x}{y}$ of the point P is $= n\frac{\rho-\alpha}{\rho-\beta}$; or in particular, if $n=1$, then the coordinate is $= \frac{\rho-\alpha}{\rho-\beta}$. If for $n\frac{\rho-\alpha}{\rho-\beta}$ we write $\frac{-\beta\rho+\lambda}{\rho-\beta}$, and then take $\beta = \infty$, we see that in a particular system of coordinates, A at O, B at ∞, the coordinate $\frac{x}{y}$ is $= \rho$. Proceeding in the same manner in regard to the coordinates (x', y'), for a particular system of coordinates, A′ at O′, B′ at ∞, the coordinate $\frac{x'}{y'}$ of P′ will be $= \rho'$. And the correspondence of the points P, P′ will be given by an equation

$$a\rho\rho' + b\rho' + c\rho + d = 0.$$

14. The equation just mentioned is often convenient for obtaining a precise statement of theorems. Thus taking A, B at pleasure on the first line, A′, B′ the corresponding points on the second line, we have

$$\rho' = -\frac{c\rho+d}{a\rho+b}, \text{ and thence}$$

$$\alpha' = -\frac{c\alpha+d}{a\alpha+b}, \qquad \rho'-\alpha' = \frac{(ad-bc)(\rho-\alpha)}{(a\alpha+b)(a\rho+b)},$$

$$\beta' = -\frac{c\beta+d}{a\beta+d}, \qquad \rho'-\beta' = \frac{(ad-bc)(\rho-\beta)}{(a\beta+b)(a\rho+b)};$$

and consequently $$\frac{\rho'-\alpha'}{\rho'-\beta'} = \frac{a\beta+b}{a\alpha+b}\frac{\rho-\alpha}{\rho-\beta},$$

which is of the form $$\frac{\rho'-\alpha'}{\rho'-\beta'} = m\frac{\rho-\alpha}{\rho-\beta},$$

where (the correspondence $a\rho\rho' + b\rho' + c\rho + d = 0$ being given, and also the fixed points A, B) m has a determinate value not assumable at pleasure. If, however, the fixed points A, B be not given, then we may determine a relation between them, such that m shall have any given value not being $= 1$; we have in fact only to write

$$a\beta + b = m(a\alpha + b),$$

that is $$a(\beta - m\alpha) + b(1-m) = 0.$$

($m=1$ would give $\alpha=\beta$ and the transformation would fail.) In particular we may write $m=-1$, we have then

$$a(\alpha+\beta) + 2b = 0;$$

or the sum of the two distances OA, OB has a given value $= -\frac{2b}{a}$ dependent on the transformation; one of these points being assumed at pleasure, the other is known; the points A′, B′ are also known, and the equation of correspondence is

$$\frac{\rho'-\alpha'}{\rho'-\beta'}+\frac{\rho-\alpha}{\rho-\beta}=0;$$

it is moreover easy to show that we have

$$a(\alpha'+\beta')+2c=0.$$

15. In what precedes, the two lines are considered as distinct lines, not of necessity existing in a common space. But they may be considered, not only as existing in the common space, but as superimposed the one on the other. Suppose this is so, and moreover that the fixed points A′, B′ coincide with A, B respectively, and that the coordinates (x, y) and (x', y') are the same coordinates; so that the equation $xy'-x'y=0$ will imply the coincidence of the points P, P′.

16. If $ad-bc=0$, the equation of correspondence becomes

$$(ax+by)(a'x'+b'y')=0,$$

and as before, to a single given point $ax+by=0$, considered as belonging to the first figure, there corresponds every point whatever of the line, or second figure: to a single given point $a'x'+b'y'=0$ (the same as, or different from, the first point), considered as belonging to the second figure, there corresponds every point whatever of the line, or first figure.

17. Excluding the foregoing case, or assuming $ad-bc\neq 0$, there are in general on the line two points such that to each of them considered as belonging to either figure there corresponds the same point considered as belonging to the other figure, or say there are two united points: in fact, writing $x':y'=x:y$, we find $ax^2+(b+c)xy+dy^2=0$, a quadric equation for the determination of the points in question. Unless $4ad-(b+c)^2=0$, this equation will have two unequal roots; and taking the two points so determined for the fixed points A=A′, B=B′, the equation of correspondence will assume the form $xy'-kx'y=0$. In this equation k cannot be $=1$; for if it were so, the equation would be $xy'-x'y=0$; that is, the points P, P′ would be always one and the same point. The equation may, however, be $xy'+x'y=0$; the points P, P′ are then harmonics in regard to the fixed points A, B. It is to be observed, that if the equation $xy'-kx'y=0$ be unaltered by the interchange of (x, y) and (x', y') we must have $k^2-1=0$, or since $=1$ is excluded, we must have $k=-1$.

18. The original equation $(ax+by)x'+(cx+dy)y'=0$ is unaltered by the interchange only if $b-c=0$; the equation $4ad-(b+c)^2=0$

becomes in this case $ad-bc=0$, which by hypothesis is not satisfied; the two distinct points A=A′, B=B′ consequently exist. That is, if the correspondence between the two points P, P′ is such that whether P be considered as belonging to the first figure or to the second figure, there corresponds to it in the other figure the same point P′—or say if the correspondence between the points P, P′ is a symmetrical correspondence—then as united points in the superimposed figures we have the two distinct points A, B: and the correspondence of the points P, P′ is given by the condition that these are harmonics in regard to the points A, B.

19. There is still the case to be considered where $4ad-(b+c)^2=0$; the equation $ax^2+(b+c)xy+dy^2=0$ has here equal roots, or the two united points coincide together, or form a single point. Taking this point to be the point A, the coordinate whereof is $x:y=0:1$, we must, it is clear, have $d=0$, and therefore also $b+c=0$: the relation between the coordinates (x, y) and (x', y') is then $axx'+b(xy'-x'y)=0$; viz., this is the form assumed by the equation of correspondence when instead of two united points there is a double united point, and this is taken to be the fixed point A.

20. It is to be observed, that we cannot have either $b=0$, for this would give $xx'=0$, which belongs to the excluded case $ad-bc=0$; nor $a=0$, for this would give $xy'-x'y=0$: excluding these cases, the equation is of necessity altered by the interchange of (x, y) and (x', y'); that is, in the case of a double united point, the transformation is essentially unsymmetrical.

By what precedes, if the other fixed point be taken to be at infinity, the coordinates $x:y$ and $x':y'$ may be taken to be ρ, ρ' respectively; viz., ρ, ρ' will denote the distances of the points P, P′ from the double united point A; and the equation of correspondence then becomes $\rho\rho'+b(\rho-\rho')=0$; that is, $(\rho-b)(\rho'+b)+b^2=0$.

21. The original equation $axx'+byx'+cxy'+dyy'=0$ can be reduced to the inverse form $xx'-yy'=0$ only (it is clear) in the symmetrical case $b=c$; here, transforming to the united points, the equation is, by what precedes (*ante*, No. 17) $xy'+x'y=0$, which equation can be written $(lx+my)(lx'+my')-(lx-my)(lx'-my')=0$, where $l:m$ is arbitrary; viz., we have thus an equation of the required form.

22. In further explanation, start from the equation $a\rho\rho'+b(\rho+\rho')+d=0$; that is, $(a\rho+b)(a\rho'+b)+ad-b^2=0$, or say $(\rho-\alpha)(\rho'-\alpha)-k^2=0$; this may be reduced to $\rho\rho'-1=0$; viz., the point O from which are measured the distances ρ, ρ' is here the mid-point between the two united points A, B; and the unit of distance is $\frac{1}{2}$AB; the equation expresses that the points P, P′, harmonics in regard to the two points A, B, are the images one of the other in regard to the circle

described upon AB as diameter. Take any two corresponding points L, L′; if the distances of these be λ, λ', we have $\lambda\lambda' = 1$; and hence

$$(\rho-\lambda)(\rho'-\lambda) = 1-\lambda(\rho+\rho')+\lambda^2 = \lambda(\lambda+\lambda'-\rho-\rho'),$$
$$(\rho-\lambda')(\rho'-\lambda') = 1-\lambda'(\rho+\rho')+\lambda'^2 = \lambda'(\lambda+\lambda'-\rho-\rho');$$

and consequently

$$\frac{\rho-\lambda}{\rho-\lambda'} \cdot \frac{\rho'-\lambda}{\rho'-\lambda'} = \frac{\lambda}{\lambda'};$$

which, writing

$$\frac{x}{y} = \frac{k(\rho-\lambda)}{\rho-\lambda'}, \quad \frac{x'}{y'} = \frac{k(\rho'-\lambda)}{\rho'-\lambda'},$$

$$k^2 = \frac{\lambda}{\lambda'} \text{ (so that } k^2 \neq 1); \text{ or, } k = \lambda = \frac{1}{\lambda'},$$

becomes $xx'-yy'=0$; that is, the correspondence of the points P, P′ being symmetrical, if the coordinate $\frac{x}{y}$ of P be taken to be a multiple of the ratio of the distances PL, PL′ of P from any two corresponding points L, L′ (and of course the coordinate $\frac{x'}{y'}$ of P′ to be the same multiple of the ratio of the distances P′L, P′L′), the equation of correspondence is obtained in the inverse form $xx'-yy'=0$.

The Rational Transformation between Two Planes.

23. Starting from the equations $x' : y' : z' = X : Y : Z$, where $X=0$, $Y=0$, $Z=0$ are curves in the first plane, of the same order n, it has been seen that in order that we may thence have a rational transformation between the two planes, the curves $X=0$, $Y=0$, $Z=0$ must have a common intersection of n^2-1 points, and no more; that is, they must not have a complete common intersection of n^2 points. In the case $n=2$, taking the n^2-1 points in the first plane to be any three points whatever, the condition that the curves shall be conics passing through the three points does not in anywise imply that the conics shall have a common fourth point of intersection; and we have thus a rational transformation as required; viz., the first set of equations is $x' : y' : z' = X : Y : Z$, where $X=0$, $Y=0$, $Z=0$ are conics passing through the same three points of the first plane; and as it is easy to see (but which will be subsequently shown more in detail), the second set is the similar one $x : y : z = X' : Y' : Z'$, where $X'=0$, $Y'=0$, $Z'=0$ are conics passing through the same three points in the second plane; this may be called the quadric transformation between the two planes.

24. But the like theory would not apply to the case $n=3$; if the n^2-1 points in the first plane were any eight points whatever, the cubics $X=0$, $Y=0$, $Z=0$, intersecting in these eight points, would have a common ninth point of intersection, and the transformation

would fail; and so for any higher value of n, taking at pleasure any $\frac{1}{2}n(n+3)-1$ of the n^2-1 points of the first plane, the curves $X=0$, $Y=0$, $Z=0$ of the order n passing through these $\frac{1}{2}n(n+3)-1$ points, would have in common all their remaining points of intersection, and the transformation would fail. A transformation can only be obtained by taking the n^2-1 points in such wise that these can be made to be the common intersection of the curves, and at the same time that the number of conditions imposed upon each of the curves $X=0$, $Y=0$, $Z=0$ shall be at most $=\frac{1}{2}n(n+3)-1$.

25. And this requirement may be satisfied; viz., the number of conditions may be made to be $=\frac{1}{2}n(n+3)-1$, by assuming that certain of the n^2-1 points of intersections are multiple intersections of the curves. For if we have a given point which is an α-tuple point on each of the curves $X=0$, $Y=0$, $Z=0$, then this counts for α^2 points of intersection of any two of the curves, and thus for α^2 points of the n^2-1 points: but the condition that the given point shall be on any one of the curves, say the curve $X=0$, an α-tuple point, imposes on the curve, not α^2, but only $\frac{1}{2}\alpha(\alpha+1)$ conditions: and we have in this way a reduction whereby the number of conditions for passing through the n^2-1 points can be lowered from n^2-1 to the required number $\frac{1}{2}n(n+3)-1$.

26. In particular, for $n=3$, we may for the n^2-1 points of the first plane take a point as a double point on each of the cubic curves $X=0$, $Y=0$, $Z=0$ (which therefore reckons as four points), and take any other four points. Each of the curves is determined by the conditions of having a given point for double point, and of passing through four other points; that is, by $3+4=7$ conditions; and the three cubic curves $X=0$, $Y=0$, $Z=0$ have for the common intersection the double point reckoning as four points, and the given other four points; that is, they have a common intersection of $4+4=8$ points; but this does not imply that they have a common ninth point of intersection; we have therefore a rational transformation as required; viz., the first set of equations is $x' : y' : z' = X : Y : Z$; where $X=0$, $Y=0$, $Z=0$ are cubics in the first plane having each of them a double point at the same given point, and also each passing through the same four given points; the second set of equations is $x : y : z = X' : Y' : Z'$, where $X'=0$, $Y'=0$, $Z'=0$ are like cubics in the second plane.

27. Generally suppose that the n^2-1 points in the first plane are made up of a_1 points, which are simple points; a_2 points, which are double points; a_3 points, which are triple points, ... a_{n-1} points, which are $(n-1)$tuple points ($a_{n-1}=1$ or 0), on each of the three curves; these will represent a system of n^2-1 points if only

$$a_1+4a_2+9a_3 \ldots +(n-1)^2a_{n-1}=n^2-1.$$

The number of conditions imposed on each of the curves $X=0$, $Y=0$, $Z=0$ will be $a_1+3a_2+6a_3 \ldots +\frac{1}{2}n(n-1)a_{n-1}$; for the reason presently appearing, I exclude the case of this being $<\frac{1}{2}n(n+3)-2$; and therefore assume it to be $=\frac{1}{2}n(n+3)-2$. In fact, writing

$$a_1+3a_2+6a_3 \ldots +\tfrac{1}{2}n(n-1)a_{n-1} = \tfrac{1}{2}n(n+3)-2,$$

this combined with the former equation gives

$$a_2+3a_3 \ldots +\tfrac{1}{2}(n-1)(n-2)a_{n-1} = \tfrac{1}{2}(n-1)(n-2);$$

viz., the singularities are equivalent to $\frac{1}{2}(n-1)(n-2)$ double points, that is, to the maximum number of double points of a curve of the order n; or say each of the curves $X=0$, $Y=0$, $Z=0$ is a curve of the order n having a deficiency $=0$; that is, it is a unicursal curve of the order n. Hence also, taking (a, b, c) any constant factors whatever, the curve $aX+bY+cZ=0$ is unicursal.

28. It is important to remark that the conclusion follows directly from the general notion of the rational transformation; in fact, the equation $aX+bY+cZ=0$ is satisfied if $x:y:z=X':Y':Z'$; $ax'+by'+cz'=0$. The last of these equations determines the ratios $x':y':z'$ in terms of a single parameter (*e.g.* the ratio $x':y'$), and we have then $x:y:z$ expressed as rational functions of this parameter; that is, the curve is unicursal.

29. Suppose for a moment that it was possible to have

$$a_1+3a_2+6a_3 \ldots +\tfrac{1}{2}n(n-1)a_{n-1} < \tfrac{1}{2}n(n+3)-2.$$

Combining in the same way with the first equation, it would follow that

$$a_2+3a_3 \ldots +\tfrac{1}{2}(n-1)(n-2)a_{n-1} > \tfrac{1}{2}(n-1)(n-2),$$

which would imply that the curves $X=0$, $Y=0$, $Z=0$ break up each of them into inferior curves: but more than this, the coefficients a, b, c being arbitrary, it would imply that the curve $aX+bY+cZ=0$ breaks up into inferior curves; this can only be the case if X, Y, Z have a common factor, say M; that is, if $X, Y, Z = MX_1, MY_1, MZ_1$; but we could then omit the common factor, and in place of $x':y':z'=X:Y:Z$ write $x':y':z'=X_1:Y_1:Z_1$, where $X_1=0$, $Y_1=0$, $Z_1=0$, are proper curves, not breaking up; the above supposition may therefore be excluded from consideration.

30. We have thus a transformation in which the first set of equations is $x':y':z'=X:Y:Z$, where $X=0$, $Y=0$, $Z=0$ are curves in the first plane, of the same order n, having in common $a_1, a_2 \ldots a_{n-1}$ points which are simple points, double points, ... $(n-1)$tuple points respectively on each of the curves, these numbers satisfying the conditions

$$a_1+4a_2+9a_3 \ldots +(n-1)^2 a_{n-1} = n^2-1,$$
$$a_1+3a_2+6a_3 \ldots +\tfrac{1}{2}n(n-1)a_{n-1} = \tfrac{1}{2}(n^2+3n)-2;$$

conditions which give, as above,

$$a_2+3a_3 \ldots +\tfrac{1}{2}(n-1)(n-2)\,a_{n-1} = \tfrac{1}{2}(n-1)(n-2),$$

and also $a_1+2a_2+3a_3 \ldots +(n-1)\,a_{n-1} = 3n-3;$

so that the relations between $a_1, a_2 \ldots a_{n-1}$ are given by any two of these four equations.

31. The second set of equations then is $x : y : z = X' : Y' : Z'$, where $X'=0$, $Y'=0$, $Z'=0$ are curves in the second plane, of the same order n; and it is clear that these must be curves such as those in the first plane; viz., they must have in common $a'_1, a'_2, \ldots a'_{n-1}$ points, which are simple points, double points, ... $(n-1)$tuple points respectively on each of the curves, the relations between these numbers being expressed by any two of the four equations

$$\begin{aligned}
a'_1+4a'_2+9a'_3 \ldots +(n-1)^2\,a'_{n-1} &= n^2-1,\\
a'_1+3a'_2+6a'_3 \ldots +\tfrac{1}{2}n(n-1)\,a'_{n-1} &= \tfrac{1}{2}n(n+3)-2,\\
a'_2+3a'_3 \ldots +\tfrac{1}{2}(n-1)(n-2)\,a'_{n-1} &= \tfrac{1}{2}(n-1)(n-2),\\
a'_1+2a'_2+3a'_3 \ldots +(n-1)\,a'_{n-1} &= 3n-3.
\end{aligned}$$

32. To any line $ax'+by'+cz'=0$ in the second plane there corresponds in the first plane a curve $aX+bY+cZ=0$ of the order n; and to any line $a'x+b'y+c'z=0$ in the first plane there corresponds in the second plane a curve $a'X'+b'Y'+c'Z'=0$ of the same order n; the curves $aX+bY+cZ=0$ in the first plane are, it is clear, a system, and the entire system, of curves each satisfying the conditions which have been stated in regard to the individual curves $X=0$, $Y=0$, $Z=0$, and being as already mentioned unicursal; and similarly the curves $a'X'+b'Y'+c'Z'=0$ in the second plane are a system, and the entire system, of curves each satisfying the conditions which have been stated in regard to the individual curves $X'=0$, $Y'=0$, $Z'=0$; and being also unicursal. We may say that to the lines of the second plane there corresponds in the first plane the *reseau* of curves $aX+bY+cZ=0$; and to the lines of the first plane there corresponds in the second plane the *reseau* of curves $a'X'+b'Y'+c'Z'=0$; these reseaux being systems satisfying respectively the conditions just referred to.

33. We have next to enquire what are the curves in the second plane which correspond to the $a_1+a_2 \ldots +a_{n-1}$ points of the first plane. I remark that the $a_1+a_2 \ldots +a_{n-1}$ points are termed by Cremona the *principal points* of the first plane, and the corresponding curves the *principal curves* of the second plane. But it will be more convenient to say that the $a_1+a_2 \ldots +a_{n-1}$ points are the *principal system* of the first plane, and the corresponding curves the *principal counter-system* of the second plane. And of course the $a'_1+a'_2 \ldots +a'_{n-1}$ points will be the principal system of the second plane, and the corresponding curves the principal counter-system of the first plane.

34. The Jacobian (curve) of the curves $X=0$, $Y=0$, $Z=0$ is, of course, the Jacobian of any three curves $aX+bY+cZ=0$ of the first plane, or it may be called the Jacobian of the reseau of the first plane; and similarly, the Jacobian of the curves $X'=0$, $Y'=0$, $Z'=0$ is the Jacobian of the reseau of the second plane.

35. I say that to each point a_1 of the first figure there corresponds in the second figure a line; to each point a_2 a conic; to each point a_3 a nodal cubic; ... and generally, to each point a_r a unicursal r-thic curve; the entire system of the curves corresponding to the $a_1+a_2+a_3 \ldots +a_{n-1}$ points, that is, the principal counter-system of the second plane, is thus made up of a_1 lines, a_2 conics, a_3 nodal cubics, ... a_r unicursal r-thics, a_{n-1} unicursal $(n-1)$thics. It is thus a curve of the aggregate order $a_1+2a_2+3a_3 \ldots +(n-1)\,a_{n-1} = 3n-3$; and it is in fact the Jacobian of the reseau of the second plane; as such, it passes through each point a'_1 two times, each point a'_2 five times, ... each point a'_r $3r-1$ times, ... each point a'_{n-1} $3n-4$ times.

36. The reciprocal theorem is of course true. The Jacobian of the reseau of the first plane is thus made up of a'_1 lines, a'_2 conics, a'_3 nodal cubics, ... a'_r unicursal r-thics, ... a'_{n-1} unicursal $(n-1)$thics. Calculating the Jacobian of the reseau of the first plane, we have thus the numbers $a'_1, a'_2, \ldots a'_{n-1}$, which determine the nature of the principal system of the second plane.

37. I indicate as follows the analytical proof of the theorem that to a principal point a_r of the first plane there corresponds in the second plane a unicursal r-thic. Consider the simplest case, $r=1$; if in the equations $x' : y' : z' = X : Y : Z$ the coordinates (x, y, z) are considered as belonging to a point a_1, these values give identically $X=0$, $Y=0$, $Z=0$; hence for the consecutive point $x+\delta x, y+\delta y, z+\delta z$, if (A, B, C) denote the derived functions of X, (A_1, B_1, C_1) those of Y, (A_2, B_2, C_2) those of Z, we have

$$\begin{aligned} x' : y' : z' = \quad & A\,\delta x + B\,\delta y + C\,\delta z \\ & : A_1\delta x + B_1\delta y + C_1\delta z \\ & : A_2\delta x + B_2\delta y + C_2\delta z. \end{aligned}$$

We have $\begin{vmatrix} A, & B, & C \\ A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \end{vmatrix} = 0$, for the determinant is the value, at the point a_1 in question, of the Jacobian of the reseau of the first plane: and the Jacobian curve passing through a_1 (in fact, having there a double point), the value is $=0$.

38. Hence x', y', z', considered as corresponding to a point indefinitely near to a_1, are connected by a linear equation. Corresponding to a_1 we

have in the second figure a line. But it is to be observed, further, that the equation of the line is that obtained by writing in the foregoing equations say $\delta z = 0$, and eliminating the remaining quantities δx, δy; or what is the same thing, we may consider the equation of the line as given by the equations

$$\begin{aligned} x' : y' : z' = \quad & A\,\delta x + B\,\delta y \\ & : A_1\delta x + B_1\delta y \\ & : A_2\delta x + B_2\delta y, \end{aligned}$$

where δx, δy are indeterminate parameters to be eliminated.

39. In the case of a point a_r we have in like manner

$$\begin{aligned} x' : y' : z' = \quad & (a, \ldots \mathcal{X}\delta x, \delta y, \delta z)^r \\ & : (a_1, \ldots \mathcal{X}\delta x, \delta y, \delta z)^r \\ & : (a_2, \ldots \mathcal{X}\delta x, \delta y, \delta z)^r, \end{aligned}$$

where $(a, \ldots)$, $(a_1, \ldots)$, $(a_2, \ldots)$ are the r-th derived functions of X, Y, Z respectively. In virtue of the relation of the point a_r to the curves $X = 0$, $Y = 0$, $Z = 0$, the coefficients will be such as to allow of the simultaneous elimination from these equations of the three quantities δx, δy, δz. The result of the elimination will be the same as if, writing say $\delta z = 0$, we eliminate δx, δy; or what is the same thing, the relation of x', y', z' may be regarded as given by the equations

$$\begin{aligned} x' : y' : z' = \quad & (a, \ldots \mathcal{X}\delta x, \delta y)^r \\ & : (a_1, \ldots \mathcal{X}\delta x, \delta y)^r \\ & : (a_2, \ldots \mathcal{X}\delta x, \delta y)^r, \end{aligned}$$

where δx, δy are indeterminate parameters. These equations obviously express that the point (x', y', z') is situate on a unicursal curve of the order r.

40. It is further to be shown that the r-thic curve thus corresponding to a_r is part of the Jacobian of the reseau of the second plane. The Jacobian in question is the locus of the new double point of those curves of the reseau which have a new double point; that is, a double point not included among the $a'_2 + a'_3 \ldots + a'_{n-1}$ singular points of the principal system of the second plane. But a curve of the reseau being unicursal, can only acquire a new double point by breaking up into inferior curves. Consider, in the first figure, any line through a_r, the corresponding curve in the second figure is made up of the unicursal r-thic curve, which corresponds to the point a_r, together with a residual curve variable with the line through a_r; this is a unicursal curve of the order $n-r$. The aggregate curve of the order $r+(n-r)$ has singular

points equivalent to $\frac{1}{2}(n-1)(n-2)+1$ double points;* that is, the singularities are those belonging to the principal system of the second plane, together with a new double point constituted by an intersection of the curves r, $n-r$. [Observe that the two curves have only this single intersection; viz., the remaining $r(n-r)-1$ intersections are at points $\alpha'_2+\alpha'_3 \ldots +\alpha'_{n-1}$ of the principal system of the second plane.] We have thus, in the second plane, a series of curves, each of them having a new double point; viz., these are the several curves which correspond to the lines through a_r in the first figure. Each of the curves is a fixed curve r together with a variable curve $n-r$. The new double point is an intersection of the two curves; that is, it is a variable point on the curve r. The locus of the new double point is thus the curve r; therefore the curve r is part of the Jacobian of the reseau of the second plane. Since each point a_r gives a curve r, the curves in question form an aggregate curve of the order $\alpha_1+2\alpha_2 \ldots + (n-1)\alpha_{n-1}$, $=3n-3$; viz., this is the order of the Jacobian; or, as stated, the curves r (that is, the principal counter-system of the second plane) constitute the Jacobian of the reseau of this plane.

41. The numerical systems $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ and $(\alpha'_1, \alpha'_2 \ldots \alpha'_{n-1})$ are each of them a solution of the same two indeterminate equations

$$\Sigma r^2 \alpha_r = n^2-1, \quad \Sigma r\alpha_r = 3n-3,$$

but not every solution of these equations is admissible; for instance, if $r>\frac{1}{2}n$, then α_r is $=0$ or 1, for $\alpha_r=2$ would imply a curve of the order n with two r-tuple points, and the line joining these would meet the curve in more than r points; similarly, $r>\frac{2}{5}n$, α_r is $=4$ at most, for $\alpha_r=5$ would imply a curve of the order n with five r-tuple points, and the conic through these would meet the curve in more than $2n$ points; and there are of course other like restrictions. The different admissible systems up to $n=10$ are tabulated in Cremona's Memoir; and he has also given systems belonging to certain specified forms of n: these results are as follows:—

* In general, if $r+r'=n$, and the curves r, r' are each unicursal, then the aggregate singularity arising from the singularities of the two curves and from their intersections, is equivalent to $\frac{1}{2}(r-1)(r-2)+\frac{1}{2}(r'-1)(r'-2)+rr'$, that is, to $\frac{1}{2}(r+r'-1)(r+r'-2)+1$, or $\frac{1}{2}(n-1)(n-2)+1$ double points.

n	2	3	4		5			6				7				
a_1	3	4	6	3	8	3	0	10	1	4	3	12	2	0	5	3
a_2		1	0	3	0	3	6	0	4	1	4	0	3	3	0	5
a_3			1	0	0	1	0	0	2	3	0	0	2	4	3	0
a_4					1	0	0	0	0	0	1	0	1	0	1	0
a_5								1	0	0	0	0	0	0	0	1
a_6												1	0	0	0	0
a_7																
a_8																
a_9																
	1	1	1	2	1	2	3	1	2	3	4	1	2	3	4	5

n	8									9									
a_1	14	3	1	0	3	6	0	2	3	16	4	2	0	3	7	1	3	0	1
a_2	0	2	3	0	6	0	5	0	3	0	1	3	4	7	0	4	0	3	1
a_3	0	3	2	7	0	1	2	5	0	0	4	1	0	0	0	3	4	3	3
a_4	0	0	2	0	0	3	0	1	3	0	0	2	4	0	3	0	1	1	3
a_5	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	1	1	0
a_6	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0
a_7	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_8										1	0	0	0	0	0	0	0	0	0
a_9																			
	1	2	3	4	5	6	7	8	9*	1	2	3	4	5	6	7	8	9	10

n	10																
a_1	18	5	1	0	0	3	8	2	4	1	2	3	3	3	0	0	1
a_2	0	0	4	2	0	8	0	3	0	3	1	3	3	0	6	1	0
a_3	0	5	0	2	7	0	0	4	3	2	3	0	1	0	0	5	2
a_4	0	0	2	3	0	0	1	0	2	2	1	3	0	6	0	0	5
a_5	0	0	2	1	0	0	3	0	0	0	2	0	3	0	3	2	0
a_6	0	0	0	0	1	0	0	0	1	1	0	1	0	0	0	0	0
a_7	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
a_8	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
a_9	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

* Omitted by Cremona.

$n = p$

α_1	$= 2p-2$	
α_{p-1}	$= 1$	

$n = 2p$

α_1	$= 3$	$2p-2$
α_2	$= 2p-2$	0
α_{p-1}	$= 0$	1
α_p	$= 0$	3
α_{2p-2}	$= 1$	0

$n = 2p+1$

α_1	$= 3$	$2p-1$
α_2	$= 2p-1$	0
α_p	$= 0$	3
α_{p+1}	$= 0$	1
α_{2p-1}	$= 1$	0

$n = 3p$

α_1	$= 1$	$2p-3$
α_2	$= 4$	0
α_3	$= 2p-3$	0
α_p	$= 0$	4
α_{p+1}	$= 0$	1
α_{2p-1}	$= 0$	1
α_{3p-3}	$= 1$	0

α_1	$= 4$	$2p-2$
α_2	$= 1$	0
α_3	$= 2p-2$	0
α_{p-1}	$= 0$	1
α_p	$= 0$	4
α_{2p}	$= 0$	1
α_{3p-3}	$= 1$	0

$n = 3p+1$

α_1	$= 2$	$2p-2$
α_2	$= 3$	0
α_3	$= 2p-2$	0
α_p	$= 0$	3
α_{p+1}	$= 0$	2
α_{2p}	$= 0$	1
α_{3p-2}	$= 1$	0

α_1	$= 5$	$2p-1$
α_3	$= 2p-1$	0
α_p	$= 0$	5
α_{2p+1}	$= 0$	1
α_{3p-2}	$= 1$	0

$n = 3p+2$

α_1	$= 3$	$2p-1$
α_2	$= 2$	0
α_3	$= 2p-1$	0
α_p	$= 0$	2
α_{p+1}	$= 0$	3
α_{2p+1}	$= 0$	1
α_{3p-1}	$= 1$	0

α_1	$= 0$	$2p-2$
α_2	$= 5$	0
α_3	$= 2p-2$	0
α_{p+1}	$= 0$	5
α_{2p}	$= 0$	1
α_{3p-1}	$= 1$	0

$n = 4p$

α_1	$= 1$	$2p-3$
α_2	$= 3$	0
α_3	$= 2$	0
α_4	$= 2p-3$	0
α_p	$= 0$	3
α_{p+1}	$= 0$	1
α_{2p-1}	$= 0$	1
α_{2p}	$= 0$	2
α_{4p-1}	$= 1$	0

α_1	$= 2$	$2p-4$
α_3	$= 5$	0
α_4	$= 2p-4$	0
α_p	$= 0$	5
α_{p+1}	$= 0$	2
α_{3p-1}	$= 0$	1
α_{4p-4}	$= 1$	0

α_1	$= 3$	$2p-2$
α_2	$= 3$	0
α_4	$= 2p-2$	0
α_{p-1}	$= 0$	1
α_p	$= 0$	3
α_{2p}	$= 0$	3
α_{4p-4}	$= 1$	0

α_1	$= 6$	$2p-2$
α_3	$= 1$	0
α_4	$= 2p-2$	0
α_{p-1}	$= 0$	1
α_p	$= 0$	6
α_{3p}	$= 0$	1
α_{4p-4}	$= 1$	0

$n = 4p+1$

$a_1 = 0$	$a_1 = 2$	$a_1 = 3$
$a_2 = 3$	$a_2 = 3$	$a_3 = 4$
$a_3 = 3$	$a_3 = 1$	$a_4 = 2p-3$
$a_4 = 2p-3$	$a_4 = 2p-2$	$a_p = 0$
$a_p = 0$	$a_p = 0$	$a_{p+1} = 0$
$a_{p+1} = 0$	$a_{p+1} = 0$	$a_{3p} = 0$
$a_{2p} = 0$	$a_{2p} = 0$	$a_{4p-3} = 1$
$a_{4p-3} = 1$	$a_{2p+1} = 0$	
	$a_{4p-3} = 1$	

$2p-3$	$2p-2$	$2p-3$
0	0	0
0	0	0
0	0	0
1	0	4
3	1	3
3	2	1
0	1	0
	3	
	0	

$a_1 = 7$	$2p-$
$a_4 = 2p-1$	0
$a_p = 0$	7
$a_{3p+1} = 0$	1
$a_{4p-3} = 1$	0

$n = 4p+2$

$a_1 = 0$	$a_1 = 1$	$a_1 = 3$
$a_3 = 7$	$a_2 = 3$	$a_2 = 3$
$a_4 = 2p-4$	$a_3 = 2$	$a_4 = 2p-1$
$a_{p+1} = 0$	$a_4 = 2p-2$	$a_p = 0$
$a_{3p} = 0$	$a_p = 0$	$a_{p+1} = 0$
$a_{4p-2} = 1$	$a_{p+1} = 0$	$a_{2p+1} = 0$
	$a_{2p} = 0$	$a_{4p-2} = 1$
	$a_{2p+1} = 0$	
	$a_{4p-2} = 1$	

$2p-4$	$2p-2$	$2p-1$
0	0	0
0	0	0
7	0	3
1	1	1
0	3	3
	1	0
	2	
	0	

$a_1 = 4$	$2p-$
$a_3 = 3$	0
$a_4 = 2p-2$	0
$a_p = 0$	3
$a_{p+1} = 0$	4
$a_{3p} = 0$	1
$a_{4p-2} = 1$	0

$n = 4p+3$

$a_1 = 0$	$a_1 = 9$	$a_1 = 2$
$a_2 = 3$	$a_3 = 6$	$a_2 = 3$
$a_3 = 3$	$a_4 = 2p-3$	$a_3 = 1$
$a_4 = 2p-2$	$a_{p+1} = 0$	$a_4 = 2p-1$
$a_{p+1} = 0$	$a_{p+2} = 0$	$a_p = 0$
$a_{p+2} = 0$	$a_{3p+1} = 0$	$a_{p+1} = 0$
$a_{2p+1} = 0$	$a_{4p-1} = 1$	$a_{2p+1} = 0$
$a_{4p-1} = 1$		$a_{2p+2} = 0$
		$a_{4p-1} = 1$

$2p-2$	$2p-3$	$2p-1$
0	0	0
0	0	0
0	0	0
3	6	1
1	1	3
1	1	2
3	0	1
0		0

$a_1 = 5$	$2p-$
$a_3 = 2$	0
$a_4 = 2p-1$	0
$a_p = 0$	2
$a_{p+1} = 0$	5
$a_{3p+2} = 0$	1
$a_{4p-1} = 1$	0

42. The system $(a_1, a_2 \ldots a_{n-1})$ geometrically determines completely the system $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$; it ought therefore to determine it arithmetically; that is, given the one series of numbers, we ought to be able to determine, or at least to select from the table, the other series of numbers. Cremona has shown that the two series consist of the same numbers in the same or a different order. By examination of the tables, it appears that there are certain columns which are single (that is, no other column contains in a different order the same numbers), others that occur in pairs, the two columns of a pair containing the same numbers in a different order. Where the column is single, it is clear that this must give as well the values of $(a_1, a_2 \ldots a_{n-1})$ as of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$. Where there is a pair of columns, as far as Cremona has examined, if the one column is taken to be $(a_1, a_2 \ldots a_{n-1})$ the other column is $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$; it appears, however, not to be shown that this is universally the case; viz., it is not shown but that the two columns, instead of being reckoned as a pair, might be reckoned as two separate columns, each by itself representing the values as well of $(a_1, a_2 \ldots a_{n-1})$ as of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$; neither is it shown that there are not, in any case, more than two columns having the same numbers in different orders. It seems, however, natural to suppose that the law, as exhibited in the tables, holds good generally; viz., that the tables contain only single columns, each giving the values as well of $(a_1, a_2 \ldots a_{n-1})$ as of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$; or else pairs of columns, one giving the values of $(a_1, a_2 \ldots a_{n-1})$, and the other those of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$; or, say, that the partitions are either *self-conjugate*, or else *conjugate*.

43. Assuming that the two systems $(a_1, a_2 \ldots a_{n-1})$ and $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ are each known, there is still a question of grouping to be settled, viz., the Jacobian of the first plane consists of a_1 lines, a_2 conics, ... a_{n-1} unicursal $(n-1)$-ics; each line, each conic, &c., passes a certain number of times through each of the points $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$: but through which of them? For instance, each of the a_1 lines will pass through two of the points $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$: will these be points α_1, or points α_2, &c., or a point α_1 and a point α_2, &c.? The mere symmetry of the different groups of points determines certain conditions of the solution,* for instance, if any particular one of the a_1 lines passes through two points α_1, then each of the a_1 lines must pass through two points α_1, and since the points α_1 are symmetrical, we must in this way use all the pairs of points α_1; that is, if $a_1 = \frac{1}{2}n(n-1)$, but not otherwise.

* It is by such considerations of symmetry that Cremona has demonstrated the before-mentioned theorem of the identity of the numbers $(a_1, a_2 \ldots a_{n-1})$ and $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$.

it may be that each of the α'_1 lines passes through two of the points a_r. In the case of an equality $a_r = a_s$ we could not hereby decide whether the line passed through two points a_r or through two points a_s. So, again, if any one of the α'_1 lines pass through a point a_r and a point a_s, then each of the α'_1 lines must do so likewise, and we must hereby exhaust the combinations of a point a_r with a point a_s; viz., the assumed relation can only hold good if $\alpha'_1 = a_r a_s$. Similarly, each of the α'_2 conics will pass through five of the points $a_1, a_2 \ldots a_{n-1}$; each of the α'_2 nodal cubics will pass twice through one (have a double point there) and through six others of the points $a_1, a_2 \ldots a_{n-1}$: which are the points so passed through? I do not know how a general solution is to be obtained, but most of the cases within the limits of the foregoing table have been investigated by Cremona. The results may conveniently be stated in a tabular form; the tables exhibit in the outside upper line the values of $a_1, a_2 \ldots a_{n-1}$, and in the outside left-hand line the values of $\alpha'_1, \alpha'_2 \ldots \alpha'_{n-1}$, and they are to be read as follows: Each of the

$$\left\{\begin{matrix} \alpha'_1 \text{ lines} \\ \alpha'_2 \text{ conics} \\ \&c. \end{matrix}\right\} \text{ passes } (\ ) \text{ times through } (\ ) \text{ of the points } a_1, a_2 \ldots a_{n-1}$$

respectively; the numbers in the table being those of the points passed through, and the indices in the table (index = 1 when no index is expressed) showing the number of lines of passage, that is, showing whether the point is a simple, double, triple, &c., point on the curve referred to.

44. Thus the last of the tables $n = 6$ gives the constitution of the Jacobian of the first plane, where the principal system is (3, 4, 0, 1, 0); and it is to be read:—

Each of the 4 lines passes through 1 of the points a_1 and through the point a_4;

The 1 conic „ „ 4 of the points a_2 and through the point a_4;

Each of the 3 cubics „ „ 2 of the points a_1, 4 of the points a_2, and twice through the point a_4 (that is, a_4 is a double point on each cubic).

It is hardly necessary to remark that the tables are sibi-reciprocal, or else conjugate, as appears by the outer lines of each table.

	α_1 = 3
$\alpha'_1 = 3$	3

TABLE $n = 2$.

	α_1 = 4	α_2 = 1
$\alpha'_1 = 4$	1	1
$\alpha'_2 = 1$	4	1

TABLE $n = 3$.

	α_1 = 6	α_2 = 0	α_3 = 1
$\alpha'_1 = 6$	1		1
$\alpha'_2 = 0$			
$\alpha'_3 = 1$	6		1^2

	α_1 = 3	α_2 = 3	α_3 = 0
3		2	
3	2	3	
0			

TABLES $n = 4$.

TABLES $n = 5$.

	α_1 = 8	α_2 = 0	α_3 = 0	α_4 = 1
$\alpha_1 = 8$	1			1
$\alpha'_2 = 0$				
$\alpha'_3 = 0$				
$\alpha'_4 = 1$	8			1^3

	α_1 = 3	α_2 = 3	α_3 = 1	α_4 = 0
3		1	1	
3	1	3	1	
1	3	3	1^2	
0				

	α_1 = 0	α_2 = 6	α_3 = 0	α_4 = 0
0				
6		5		
0				
0				

TABLES $n = 6$.

	a_1 = 10	a_2 = 0	a_3 = 0	a_4 = 0	a_5 = 1
$a_1' = 10$	1				1
$a_2' = 0$					
$a_3' = 0$					
$a_4' = 0$					
$a_5' = 1$	10				1^4

	a_1 = 1	a_2 = 4	a_3 = 2	a_4 = 0	a_5 = 0
1			2		
4		3	2		
2	1	4	$1_2^2 1$*		
0					
0					

	a_1 = 4	a_2 = 1	a_3 = 3	a_4 = 0	a_5 = 0
$a_1' = 3$			2		
$a_2' = 4$	1	1	3		
$a_3' = 0$					
$a_4' = 1$	4	1	3^2		
$a_5' = 0$					

	a_1 = 3	a_2 = 4	a_3 = 0	a_4 = 1	a_5 = 0
4	1			1	
1		4		1	
3	2	4		1^2	
0					
0					

* Read, "Each of the two cubics passes through the point a_1, the four points a_2, and, $(1^2, 1)$, twice through one and once through the other of the points a_2."

45. It is to be remarked upon the tables—first, as regards the lines: if we add the numbers in each line, reckoning m^p as mp, (that is, each multiple point, according to the number of branches through it,) the sums for the successive lines are 2, 5, 8, 11, 14, &c.; that is, each line passes through 2 points, each conic through 5 points, each cubic through 8 points, each quartic through 11 points, &c. But if we add the numbers reckoning m^p as $m \cdot \frac{1}{2}p(p+1)$, (that is, each multiple point according to its effect in the determination of the curve,) then the sums are 2, 5, 9, 14, 20, &c., that is, all the curves are completely determined, viz., the line by 2 conditions, the conic by 5 conditions, the cubic by 9 conditions, &c. Secondly, as regards the columns, if for any column, reckoning m^p as mp, we multiply each number by the corresponding outside left-hand number, add, and divide the sum by the outside number at the head of the column, the successive results are

2, 5, 8, 11, 14, &c.; this merely expresses the known circumstance that the Jacobian passes $3r-1$ times through each point a_r.

46. The analogous tables showing the passage of the Jacobian through the principal system, in the solutions belonging to certain special forms of n, are

TABLE $n = p$.

	a_1	a_{p-1}
	$2p-2$	1
$a'_1 = 2p-2$	1	1
$a'_{p-1} = 1$	$2p-2$	1^{p-2}

TABLES $n = 2p$.

	a_1	a_2	a_{p-1}	a_p	a_{2p-2}
	3	$2p-2$	0	0	1
$a'_1 = 2p-2$	1				1
$a'_2 = 0$					
$a'_{p-1} = 1$		$2p-2$			1^{p-2}
$a'_p = 3$	2	$2p-2$			1^{p-1}
$a'_{2p-2} = 0$					

	a_1	a_2	a_{p-1}	a_p	a_{2p-2}
	$2p-2$	0	1	3	0
$a'_1 = 3$				2	
$a'_2 = 2p-2$	1		1	3	
$a'_{p-1} = 0$					
$a'_p = 0$					
$a'_{2p-2} = 1$	$2p-2$		1^{p-2}	3^{p-1}	

TABLES $n = 2p+1$.

	a_1	a_2	a_p	a_{p+1}	a_{2p-1}
	3	$2p-1$	0	0	1
$a'_1 = 2p-1$		1			1
$a'_2 = 0$					
$a'_p = 3$	1	$2p-1$			1^{p-1}
$a'_{p+1} = 1$	3	$2p-1$			1^p
$a'_{2p-1} = 0$					

	a_1	a_2	a_p	a_{p+1}	a_{2p-1}
	$2p-1$	0	3	1	0
$a'_1 = 3$	1			1	
$a'_2 = 2p-1$	1		3	1	
$a'_p = 0$					
$a'_{p+1} = 0$					
$a'_{2p-1} = 1$	$2p-1$		3^{p-1}	1^p	

TABLES $n = 3p$.

	a_1	a_2	a_3	a_p	a_{p+1}	a_{2p-1}	a_{3p-3}
	1	4	$2p-3$	0	0	0	1
$a'_1 = 2p-3$			1				1
$a'_2 = 0$							
$a'_3 = 0$							
$a'_p = 4$		3	$2p-3$				1^{p-1}
$a'_{p+1} = 1$	1	4	$2p-3$				1^p
$a'_{2p-1} = 1$	1	4	$(2p-3)^2$				1^{2p-3}
$a'_{3p-3} = 0$							

	a_1	a_2	a_3	a_p	a_{p+1}	a_{2p-1}	a_{3p-3}
	$2p-3$	0	0	4	1	1	0
$a'_1 = 1$					1	1	
$a'_2 = 4$				3	1	1	
$a'_3 = 2p-3$	1			4	1	1^2	
$a'_p = 0$							
$a'_{p+1} = 0$							
$a'_{2p-1} = 0$							
$a'_{3p-3} = 1$	$2p-3$			4^{p-1}	1^p	1^{2p-3}	

TABLES $n = 3p$.

	α_1 = 4	α_2 = 1	α_3 = $2p-2$	α_{p-1} = 0	α_p = 0	α_{2p} = 0	α_{3p-3} = 1
$\alpha'_1 = 2p-2$			1				1
$\alpha'_2 = 0$							
$\alpha'_3 = 0$							
$\alpha'_{p-1} = 1$			$2p-2$				1^{p-2}
$\alpha'_p = 4$	1	1	$2p-2$				1^{p-1}
$\alpha'_{2p} = 1$	4	1	$(2p-2)^2$				1^{2p-2}
$\alpha'_{3p-3} = 0$							

	α_1 = $2p-2$	α_2 = 0	α_3 = 0	α_{p-1} = 1	α_p = 4	α_{2p} = 1	α_{3p-3} = 0
$\alpha'_1 = 4$					1	1	
$\alpha'_2 = 1$					4	1	
$\alpha'_3 = 2p-2$	1			1	4	1^2	
$\alpha'_{p-1} = 0$							
$\alpha'_p = 0$							
$\alpha'_{2p} = 0$							
$\alpha'_{3p} = 1$	$2p-2$			1^{p-2}	1^{p-1}	1^{2p-2}	

47. The before mentioned theorem, that $(a_1, a_2 \ldots a_{n-1})$ and $(a'_1, a'_2 \ldots a'_{n-1})$ are the same series of numbers, of course implies $\Sigma a_r = \Sigma a'_r$; this relation Cremona demonstrates independently, by consideration of the pencil of curves $(aX+bY+cZ)+\theta(a_1X+b_1Y+c_1Z)=0$, ($\theta$ a variable parameter,) which corresponds in the first plane to the pencil of lines $(ax'+by'+cz')+\theta(a_1x'+b_1y'+c_1z')=0$, which pass through a fixed point $(ax'+by'+cz'=0,\ a_1x'+b_1y'+c_1z'=0)$ in the second figure. In general, in the pencil $U+\theta V=0$ (U, V given functions of the order n) there are $3(n-1)^2$ values of θ, each giving a nodal curve. But in the present case each of the curves $U=0$, $V=0$ has multiple points at the principal points a_r of the first plane: the question is to obtain the number of values which give a curve having one new double point; and this is found to be $= 3(n-1)^2 - \Sigma(r-1)(3r+1)a_r$. We have $\Sigma r^2 a_r = n^2-1$, $\Sigma r a_r = 3n-3$; or, substituting, the value of θ is $= \Sigma a_r$. But the curves which have an additional double point are those which correspond to the lines which in the second figure pass through one of the principal points a'_r; viz., these are the lines drawn from the point $(ax'+by'+cz'=0,\ a_1x'+b_1y'+c_1z'=0)$ to the several principal points a'_r; and the number of them is $= \Sigma a'_r$. We have thus the required relation $\Sigma a_r = \Sigma a'_r$.

The Quadric Transformation between Two Planes.

48. This is of course given by what precedes. The principal system in each plane is a set of three points; and the Jacobian of the same plane is the set of three lines joining each pair of points; that is, the three lines of either plane are the principal counter-system of the other plane. But to give the analytical investigation directly: taking the coordinates (x, y, z) to refer to the principal system of the first plane (viz., taking the three points to be the vertices of the triangle formed by the lines $x=0$, $y=0$, $z=0$), then $X=0$, $Y=0$, $Z=0$ being conics through the three points, the functions X, Y, Z will be each of them of the form $fyz+gzx+hxy$; x', y', z' being proportional to three such functions, there will be linear functions of x', y', z' proportional to yz, zy, xy; or taking these linear functions of the original (x', y', z') for the coordinates (x', y', z') of a point in the second plane, the formulæ of transformation will be $x' : y' : z' = yz : zx : xy$, and we have then conversely $x : y : z = y'z' : z'x' : x'y'$; that is, the formulæ for the transformation in question are

$$x' : y' : z' = yz : zx : xy, \text{ and } x : y : z = y'z' : z'x' : x'y'.$$

We at once verify *à posteriori* that the Jacobian in the first plane is $xyz=0$, and that in the second plane is $x'y'z'=0$.

The equations may be written

$$x' : y' : z' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}, \text{ and } x : y : z : = \frac{1}{x'} : \frac{1}{y'} : \frac{1}{z'},$$

or, if we please, $xx' = yy' = zz'$: the transformation is thus given as an inverse transformation.

[49. With respect to the metrical interpretation and actual construction of the transformation, it is to be observed that if x, y, z be taken to be proportional (not to given multiples of the perpendicular distances, but) to the perpendicular distances of P from the sides of the triangle in the first plane, and similarly x', y', z' to be proportional to the perpendicular distances of P' from the sides of the triangle in the second plane, then in general the equations of transformation must be written, not as above, but in the form $\frac{xx'}{f} = \frac{yy'}{g} = \frac{zz'}{h}$, involving arbitrary multipliers $f : g : h$. We may imagine in the second plane a point P'' determined by coordinates (x'', y'', z''),—the same coordinates as (x, y, z), that is, proportional to the perpendicular distances of P from the sides of the triangle in the second plane,—which point P'' corresponds homographically to P in such wise that $\frac{x}{f} : \frac{y}{g} : \frac{z}{h} = x'' : y'' : z''$. We have then, in the second plane, the two points P', P'' corresponding to each other in such wise that $x'x'' = y'y'' = z'z''$; and either of these points being given, the other can at once be constructed; viz., it is obvious that, joining P', P'' with any vertex, say A', of the triangle A'B'C', the lines A'P', A'P'' are equally inclined to the bisectors of the angle A'; and consequently, P'' being given, we have the three lines A'P'', B'P'', C'P'' intersecting in a common point P', which is therefore determined by means of any two of these lines. We have thus a geometrical construction of the transformation between P and P'.]

50. The analysis assumes that the principal points A, B, C of the first figure are three distinct points; but they may two of them, or all three, coincide. In the first case, say if B, C coincide, the line BC is still to be regarded as having a definite direction; and taking $x=0$ for this line, $y=0$ for the line joining A with (BC), and $z=0$ an arbitrary line through A, the functions X, Y, Z will be each of them of the form $by^2+2fyz+2hxy$; and replacing, as before, the original x, y, z by linear functions of these quantities, these linear functions being taken for the coordinates (x', y', z'), we may write $x' : y' : z' = y^2 : xy : xz$. Forming the converse system, the equations for the transformation are

$$x : y : z = y'^2 : x'y' : x'z', \text{ and } x : y : z = y'^2 : x'y' : x'z',$$

so that the points A', B', C' in the second plane are related as the points in the first plane; viz., B', C' coincide, the line B'C' being definite.

It is easy to verify that the Jacobian in the first plane is $xy'^2 = 0$, and the Jacobian in the second plane is $x'y'^2 = 0$.

51. Secondly, if A, B, C all coincide, these being however consecutive

points on a curve of finite curvature, or say on a conic; then, taking $x=0$ for the tangent at (ABC), $z=0$ for any other tangent, and $y=0$ for the chord of contact, the functions X, Y, Z will be of the form $ax^2+b(y^2-xz)+2hxy$; whence we may write $x' : y' : z' = x^2 : xy : y^2-xz$. Forming the converse equations, the equations of transformation are

$$x' : y' : z' = x^2 : xy : y^2-xz, \text{ and } x : y : z = x'^2 : x'y' : y'^2-x'z';$$

so that the points A′, B′, C′ in the second plane are related as those of the first plane; viz., they are the consecutive points of a curve of continuous curvature.

We may verify that the Jacobian of the first plane is $x^2=0$, and the Jacobian of the second plane $x'^3=0$.

The Lineo-linear Transformation between two Planes.

52. We have two equations of the form

$$(a, \ldots \S x, y, z \S x', y', z') = 0,$$
$$(a_1, \ldots \S x, y, z \S x', y', z') = 0;$$

writing these in the form

$$\mathrm{P}x'+\mathrm{Q}y'+\mathrm{R}z' = 0,$$
$$\mathrm{P}_1x'+\mathrm{Q}_1y'+\mathrm{R}_1z' = 0,$$

where $(\mathrm{P}, \mathrm{Q}, \mathrm{R}, \mathrm{P}_1, \mathrm{Q}_1, \mathrm{R}_1)$ are linear functions of (x, y, z), we have

$$x', y', z' \text{ proportional to } \left\| \begin{matrix} \mathrm{P}, & \mathrm{Q}, & \mathrm{R} \\ \mathrm{P}_1, & \mathrm{Q}_1, & \mathrm{R}_1 \end{matrix} \right\|,$$

that is to X : Y : Z, where X = 0, Y = 0, Z = 0 are conics each passing through the same three points in the first plane.

And conversely, writing the equations in the form

$$\mathrm{P}'x+\mathrm{Q}'y+\mathrm{R}'z = 0,$$
$$\mathrm{P}'_1x+\mathrm{Q}'_1y+\mathrm{R}'_1z = 0,$$

where $(\mathrm{P}', \mathrm{Q}', \mathrm{R}', \mathrm{P}'_1, \mathrm{Q}'_1, \mathrm{R}'_1)$ are linear functions of (x', y', z'), we have

$$x, y, z \text{ proportional to } \left\| \begin{matrix} \mathrm{P}', & \mathrm{Q}', & \mathrm{R}' \\ \mathrm{P}'_1, & \mathrm{Q}'_1, & \mathrm{R}'_1 \end{matrix} \right\|,$$

that is to X′, Y′, Z′, where X′ = 0, Y′ = 0, Z′ = 0 are conics each passing through the same three points in the second plane.

53. The lineo-linear transformation is thus the same thing as the quadric transformation. It is, moreover, clear that the equations must, by linear transformations on the two sets of variables respectively, and by linear combination of the two equations, be reducible into forms giving the before-mentioned values of $x : y : z$ and $x' : y' : z'$ respectively. Thus, in the general case, where in each plane the three points are distinct points, the lineo-linear equations will be reducible to

$$xx'-yy' = 0, \quad xx'-zz' = 0;$$

in the case where B, C in the first plane, and B′, C′ in the second plane respectively coincide, the forms will be

$$xx'-yy'=0,\quad yz'-y'z=0;$$

and in the case where A, B, C in the first plane, and A′, B′, C′ in the second plane respectively coincide, the forms will be

$$xy'-yx'=0,\quad xz'-yy'+zx'=0.$$

The determination of the actual formulæ for these reductions would, it is probable, give rise to investigations of considerable interest.

The General Rational Transformation between Two Planes (resumed).

54. Consider, as above, the first plane or figure with a principal system $(a_1, a_2 \ldots a_{n-1})$, and the second plane or figure with a principal system $(a'_1, a'_2 \ldots a'_{n-1})$. To a line in the second plane there corresponds in the first plane a curve of the order n passing 1 time through each of the points a_1, 2 times through each of the points a_2, 3 times through each of the points a_3, &c.; or, as we may write this,

First figure.						Second figure.					
Points	a_1	a_2	a_3	…	a_{n-1}	Points	a'_1	a'_2	a'_3	…	a'_{n-1}
	1	2	3		$n-1$		0	0	0		$n-1$
	1	2	3		$n-1$		0	0	0		$n-1$
	1	⋮	3		$n-1$		⋮	⋮	0		⋮
	⋮		⋮		⋮				⋮		
	curve order n						curve order 1				

viz., the 1's denote the number of times which the curve of the order n passes through the several points a_1 respectively; the 2's the number of times which the curve passes through the several points a_2 respectively; and so on.

55. We may, in the second figure, in the place of a line consider a curve of the order k'. If the equation hereof is $(*\,\mathfrak{X}\,x', y', z')^{k'}=0$, then the corresponding curve in the first figure is $(*\,\mathfrak{X}\,X, Y, Z)^{k'}=0$; viz., this is a curve of the order $k=nk'$. If, however, the curve in the second figure passes once or more times through all or any of the points $a'_1, a'_2 \ldots a'_{n-1}$, then there will be a depression in the order of the corresponding curve in the first figure; and, moreover, this curve will pass a certain number of times through all or some of the points $a_1, a_2, a_3 \ldots a_{n-1}$. The diagram of the correspondence will be

First figure.					Second figure.				
a_1,	a_2,	a_3	…	a_{n-1}	a'_1,	a'_2,	a'_3	…	a'_{n-1}
a_1	a_2	a_3		a_{n-1}	a'_1	a'_2	a'_3		a'_{n-1}
b_1	b_2	b_3		b_{n-1}	b'_1	b'_2	b'_3		b'_{n-1}
c_1	c_2	⋮		⋮	c'_1	⋮	⋮		c'_{n-1}
⋮	⋮				⋮				⋮
curve order k					curve order k'				

where $a_1, b_1, c_1 \ldots$ denote the number of times that the curve of the order k passes through the several points α_1 respectively, (viz., the number of the letters $a_1, b_1, c_1 \ldots$ is $= \alpha_1$, any or all of them being zeros,) $a_2, b_2, c_2 \ldots$ the number of times that the curve passes through the several points α_2 respectively, (viz., the number of the letters $a_2, b_2, c_2 \ldots$ is $= \alpha_2$, any or all of them being zeros,) and so on; and the like for the curve in the second figure.

56. By what precedes, it is easy to see that, if the curve k' passes through a point α'_1, then the curve k throws off a line, and the depression of order is $= 1$; so, if the curve passes 2 times, 3 times, ... or α'_1 times through the point in question, then the curve throws off the line repeated 2 times, 3 times, ... α'_1 times, or the depression of order is $= 2, 3 \ldots$ or α'_1; and the like for each of the points α'_1; so that, writing for shortness $a'_1 + b'_1 + c'_1 + \ldots = \Sigma a'_1$, the depression of order on account of the passages through the several points α_1 is $= \Sigma a'_1$. Similarly, for each time of passage through a point α'_2, there is thrown off a conic; or if $a'_2 + b'_2 + \ldots = \Sigma a'_2$, then the depression of order is $= 2\Sigma a'_2$, and so on; and the like for the figure in the other plane; and we thus arrive at the equations

$$k = k'n - \Sigma\,(a'_1 + 2a'_2 + 3a'_3 \ldots + \overline{n-1}\, a'_{n-1})$$
$$k' = kn - \Sigma\,(a_1 + 2a_2 + 3a_3 \ldots + \overline{n-1}\, a_{n-1}).$$

57. The simplest case is when the curve k' does not pass through any of the points $\alpha'_1, \alpha'_2, \ldots \alpha'_{n-1}$. We have then

$$a' = b'_1 = c'_1 \ldots = 0, \quad a'_2 = b'_2 \ldots = 0, \quad \ldots\ldots \quad a'_{n-1} = b'_{n-1} \ldots = 0;$$

consequently $k = k'n$. And, moreover, it is easy to see that

$$a_1 = b_1 \ldots = k', \quad a_2 = b_2 \ldots = 2k', \quad \ldots\ldots \quad a_{n-1} = b_{n-1} = \ldots = (n-1)\,k';$$

so that the correspondence is

First Figure.					Second Figure.				
a_1,	a_2,	a_3,	...	a_{n-1}	a'_1,	a_2,	a'_3,	...	a'_{n-1}	
k'	$2k'$	$3k'$		$(n-1)\,k'$	0	0	0		0	curve
k'	$2k'$	$3k'$		$(n-1)\,k'$	0	0	0		0	order
⋮	⋮	⋮		⋮	⋮	⋮	⋮		⋮	k'.

curve order $k = nk'$

We have

$$\Sigma a_1 = k'a_1, \quad \Sigma a_2 = 2k'a_2, \quad \ldots\ldots \quad \Sigma a_1 (a_1 - 1) = k'(k'-1)\,a_1, \quad \&c.,$$

and the formulæ for k, k' become

$$k = k'n, \quad k' = kn - k'\{a_1 + 4a_2 \ldots + (n-1)^2 a_{n-1}\};$$

viz., the second equation is here $k' = kn - k'(n^2-1)$; that is, $k'n^2 = kn$, agreeing, as it should do, with the first equation.

58. Moreover, the deficiency-relation is

$$\tfrac{1}{2}(k-1)(k-2) - \Sigma\,\tfrac{1}{2}\{k'(k'-1) + 2k'(2k'-1) \ldots + \overline{n-1}\,k'(\overline{n-1}\,k'-1)\}$$
$$= \tfrac{1}{2}(k'-1)(k'-2);$$

or, what is the same thing, this is

$$(nk'-1)(nk'-2) - (k'-1)(k'-2) = k'^2\{a_1 + 4a_2 \dots + (n-1)^2 a_{n-1}\} - k'\{a_1 + 2a_2 \dots + (n-1)\,a_{n-1}\}.$$

The right-hand side is

$$k'^2(n^2-1) - k'(3n-3) = (n-1)\{(n+1)k'^2 - 3k'\},$$

and we have thus the identical equation

$$(nk'-1)(nk'-2) - (k'-1)(k'-2) = (n-1)k'\{(n+1)k' - 3\}.$$

59. It should be possible, when the nature of the correspondence between the two planes is completely given, to express each of the numbers $a_1, b_1, c_1, \dots a_{n-1}, b_{n-1}, \dots$ in terms of $k, a_1, b_1, c_1, \dots a_{n-1}, b_{n-1}, \dots$; and reciprocally each of the numbers $a_1, b_1, c_1, \dots a_{n-1}, b_{n-1}, \dots$ in terms of $k, a_1, b_1, c_1, \dots a_{n-1}, b_{n-1}, \dots$; thus completing a system of relations between the two sets

$$(k, a_1, b_1, \dots a_{n-1}, b_{n-1}, \dots), \quad (k', a_1', b_1', \dots a_{n-1}', b_{n-1}', \dots);$$

but even if the theory was known, there would be considerable difficulty in forming a proper algorithm for the expression of these relations.

60. The two curves must have each of them the same deficiency. It is to be noticed, that if the curve in the first plane passes any number of times through a point P, which is not one of the points $a_1, a_2, a_3, \dots$ or a_{n-1}, then the corresponding curve in the second plane will pass the same number of times through the corresponding point P', which point will not be one of the points $a_1, a_2, \dots a_{n-1}$. The points P, P' will therefore contribute equal values to the deficiencies of the two curves respectively; so that, in equating the two deficiencies, we may disregard P, P', and attend only to the points $a_1, a_2, \dots a_{n-1}$ of the first plane, and $a_1, a_2, \dots a_{n-1}$ of the second plane. And the required relation thus is

$$\tfrac{1}{2}(k-1)(k-2) - \Sigma\,\tfrac{1}{2}\{a_1(a_1-1) + a_2(a_2-1) \dots + a_{n-1}(a_{n-1}-1)\}$$
$$= \tfrac{1}{2}(k'-1)(k'-2) - \Sigma\,\tfrac{1}{2}\{a_1(a_1-1) + a_2(a_2-1) \dots + a_{n-1}(a_{n-1}-1)\}.$$

61. In the case of the quadric transformation $n=2$, we have in the first plane the three points a_1, say these are A, B, C; and in the second plane the three points a_1, say these are A', B', C'. And if in the first plane the curve of the order k passes a, b, c times through the three points respectively, and in the second plane the corresponding curve of the order k' passes a', b', c' times through the three points respectively, then it is easy to obtain

$k' = 2k - a - b - c,$	$k = 2k' - a' - b' - c',$
$a' = k - b - c,$	$a = k' - b - c,$
$b' = k - c - a,$	$b = k' - c' - a',$
$c' = k - a - b.$	$c = k' - a - b'.$

The Quadric Transformation any number of times repeated.

62. We may successively repeat the quadric transformation according to the type—

First Fig.	Second Fig.	Third Fig.	Fourth Fig.
A, B, C	A′, B′, C′		
	D′, E′, F′	D″, E″, F″	
		G″, H″, I″	G‴, H‴, I‴

viz., in the transformation between the first and second figures, the principal systems are ABC and A′B′C′ respectively; in that between the second and third figures, they are D′E′F′ and D″E″F″ respectively; in that between the third and fourth figures, they are G″H″I″ and G‴H‴I‴; and so on. And it is then easy to see that between the first and any subsequent figure we have a rational transformation of the order 2 for the second figure, 4 for the third figure, 8 for the fourth figure, and so on.

63. But to further explain the relation, we may complete the diagram, by taking, in the transformation between the second and third figures, A″, B″, C″ to correspond to A′, B′, C′; similarly, in that between the third and fourth, A‴, B‴, C‴ to correspond to A″, B″, C″; and D‴, E‴, F‴ to correspond to D″, E″, F″. And so in the transformation between the second and third figure, we may make G′, H′, I′ correspond to G″, H″, I″, and between the first and second figures make D, E, F correspond to D′, E′, F′, and G, H, I to G′, H′, I′, the diagram being thus:

First Fig.	Second Fig.	Third Fig.	Fourth Fig.
A, B, C	A′, B′, C′	A″, B″, C″	A‴, B‴, C‴
D, E, F	D′, E′, F′	D″, E″, F″	D‴, E‴, F‴
G, H, I	G′, H′, I′	G″, H″, I″	G‴, H‴, I‴

Observe that in the principal systems (for instance, A, B, C and A′, B′, C′) the points A, B, C correspond, not to the points A′, B′, C′, but to the lines B′C′, C′A′, A′B′ respectively; and so in the other case.

64. Consider now a line in the first figure: there corresponds hereto in the second figure a conic through the points A′, B′, C′; and to this conic there corresponds in the third figure a quartic curve passing through each of the points A″, B″, C″ once, and through each of the points D″, E″, F″ twice. And conversely, to a line in the third figure corresponds in the second figure a conic through the points D′, E′, F′; and hereto in the first figure a quartic through the points D, E, F, once

and through the points A, B, C twice; that is, we have between the first and third figures a quartic transformation wherein $\alpha_1 = \alpha_2 = 3$ and $\alpha'_1 = \alpha'_2 = 3$, or say a quartic transformation $3_1 3_2$ and $3_1 3_2$. In like manner, passing to the fourth figure, to a line in the first figure corresponds in the fourth figure an octic curve passing through A‴, B‴, C‴ once, through D‴, E‴, F‴ twice, and through G‴, H‴, I‴ four times; and conversely, to a line in the fourth figure there corresponds in the first figure an octic curve passing through the points G, H, I once, the points D, E, F twice, and the points A, B, C four times; that is, between the first and fourth figures we have an octic transformation, wherein $\alpha_1 = \alpha_2 = \alpha_4 = 3$, $\alpha'_1 = \alpha'_2 = \alpha'_4 = 3$, or say a transformation, order 8, of the form $3_1 3_2 3_4$ and $3_1 3_2 3_4$. And so between the first and fifth figures there is a transformation, order 16, of the form $3_1 3_2 3_4 3_8$ and $3_1 3_2 3_4 3_8$.

65. It is, moreover, easy to find the Jacobians or counter-systems in the several transformations respectively. Thus, in the transformation between the first and second figures, in the second figure the Jacobian consists of 3 lines such as B′C′ (viz., these are, of course, the lines B′C′, C′A′, A′B′). Hence, in the transformation between the first and third figures, the Jacobian in the third figure consists of

3 conics $\quad B''C''(D''E''F'')$,
3 lines $\quad D''E''$;

viz., one of the conics is that through the five points B″, C″, D″, E″, F″, one of the lines that pass through the two points D″, E″. And so in the fourth figure, the Jacobian consists of

3 quartics $\quad B'''C'''(D'''E'''F''')_1 (G'''H'''I''')_2$,
3 conics $\quad D'''E''' \quad (G'''H'''I''')_1$,
3 lines $\quad G'''H'''$;

viz., one of the quartics passes through B‴, C‴; through D‴, E‴, F‴ each once, and through G‴, H‴, I‴ each twice. And so in the fifth figure the Jacobian consists of

3 octics $\quad B''''C''''(D''''E''''F'''')_1 (G''''H''''I'''')_2 (J''''K''''L'''')_4$,
3 quartics $\quad D''''E'''' \quad (G''''H''''I'''')_1 (J''''K''''L'''')_2$,
3 conics $\quad G''''H'''' \quad (J''''K''''L'''')_1$,
3 lines $\quad J''''K''''$,

and so on.

66. The conditions are in each case sufficient for the determination of the curve. This depends on the numerical relation

$$4+3\{1.2+2.3+4.5+8.9 \ldots +2^\theta(2^\theta+1)\} = 2^{\theta+1}(2^{\theta+1}+3).$$

The term in {} is $\quad 1+4+16 \ldots +2^{2\theta}$
$\quad +1+2+4 \ldots +2^\theta$,

that is $$\frac{2^{2\theta+2}-1}{2^2-1}+\frac{2^{\theta+1}-1}{2-1},$$

which is $$=\tfrac{1}{3}[2^{2\theta+2}-1+3\,(2^{\theta+1}-1)]$$
$$=\tfrac{1}{3}[2^{2\theta+2}+3\,.\,2^{\theta+1}-4];$$

and the relation is thus identically true.

67. Conversely, in the transformation between the first figure and the several other figures respectively, the Jacobian of the first figure is

	3 lines AB	} for order 2, between first and second figures;
and so	3 conics DE $(ABC)_1$	} for order 4, between first
	3 lines AB	} and third figures;
3 quartics	GH $(DEF)_1$ $(ABC)_2$	} for order 8, between first and fourth figures;
3 conics	DE $(ABC)_1$	
3 lines	AB	
3 octics	JK $(GHI)_1$ $(DEF)_2$ $(ABC)_4$	} for order 16, between first and fifth figures;
3 quartics	GH $(DEF)_1$ $(ABC)^2$	
3 conics	DE $(ABC)_1$	
3 lines	AB	

and so on.

Special Cases—Reduction of the General Rational Transformation to a Series of Quadric Transformations.

68. It was remarked by Mr. Clifford that any Cremona-transformation whatever may be obtained by this method of repeated quadric transformations, if only the principal systems, instead of being completely arbitrary, are properly related to each other. To take the simplest instance; suppose that we have

First figure.	Second figure.	Third figure.
A, B, C	A′, B′, C′	B″, C″
	‖	
E, F	D′, E′, F′	D″, E″, F″

viz., in the transformation between the first and second figures, we have the principal systems ABC and A′B′C′ (arbitrary as before); but in the transformation between the second and third figures, the principal systems are D′E′F′ and D″E″F″, where D′, instead of being arbitrary, coincides with A′. And we then have B″, C″ in the third figure corresponding to B′, C′ in the second figure, and E, F in the first figure corresponding to E′, F′ in the second figure. This being so, to a line in the first figure corresponds in the second figure a conic through A′, B′, C′. But A′ = D′; viz., this conic passes through a point D′ of the principal

system of the second figure, in regard to the transformation between the second and third figures. That is, (k, a, b, c referring to the second figure, and k', a', b', c' to the third figure, $k=2$, $a=1$, $b=0$, $c=0$, and therefore $k'=3$, $a'=2$, $b'=1$, $c'=1$,) corresponding to the conic we have in the third figure a curve, order 3 (cubic curve), passing twice through D″, but once through E″ and F″ respectively; this cubic curve passes also through the points B″, C″ which correspond to B′, C′ respectively; that is, cubic passes through E″, F″, B″, C″ each 1 time

„ „ D″ 2 times;

or, corresponding to a line in the first figure, we have in the third figure a curve, order 3, passing through four fixed points each 1 time, and through one fixed point 2 times. That is, we have $n=3$, $a_1'=4$, $a_2'=1$. And in the same manner, to a line in the third figure there corresponds in the first figure a cubic through four fixed points (viz., B, C, E, F) each 1 time, and through one fixed point, A, 2 times; so that also $a_1=4$, $a_2=1$. The transformation is thus of the order 3, and the form $4_1 1_2$ and $4_1 1_2$ (this is in fact the only cubic transformation; see the Tables, *ante*, No. 41).

69. Mr. Clifford has also devised a very convenient algorithm for this decomposition of a transformation of any order into quadric transformations. The quadric transformation is denoted by [3], the cubic transformation by [41], the quartic transformations by [601], [330], the quintic ones by [8001], [3310], [0600], and so on; see the Tables just referred to. (This is substantially the same as a notation employed above, the zeros enabling the omission of the suffixes; viz., $[8001]=8_1 1_4$; and so in other cases.)

70. The foregoing result is represented thus, $[4,1]=[3\overline{\wedge}0,0,1)$, which I proceed to explain. Consider in the first figure a line; the symbol [3] denotes that in the second figure we have a conic with three points (a_1'). We are about to apply to this a quadric transformation; (0, 0, 0) would denote that the three points of the principal system in the second figure were all of them arbitrary; (0, 0, 1) that one of these points was a point a_1'; (0, 1, 1) that two of them were points a_1'; (1, 1, 1) that all three of them were points a_1'; (0, 0, 2) would denote that one of the points was a point a_2'; only in the present case we can have no such symbol, by reason that there are no points a_2'. Hence $[3\overline{\wedge}001)$ denotes that the conic has applied to it a quadric transformation such that, in the transformation thereof, one point of the principal system coincides with one of the points (a_1') on the conic. To [3], *quâ* quadric transformation, belongs the number 2; and from 2, (001) we derive 3, (112), [in general k, (a, b, c) gives k', (a', b', c'), where $k'=2k-a-b-c$, $a'=k-b-c$, $b'=k-c-a$, $c'=k-a-b$]. $k=2$ corresponds to a symbol [3] of one number, $k'=3$ to a symbol of two numbers; viz., we change [3] into [30]:

we then, in the symbols (112) and (001), consider the frequencies of the several numbers 1, 2 ... taking those in the first symbol as positive, and those in the second symbol as negative; or, what is the same thing, representing the frequency as an index, we have $1^2 2^1$, 1^{-1}; or, combining, $1^{2-1} 2^1$; these indices are then added on to the numbers of [30]; viz., the index of 1 to the first number, the index of 2 to the second number (and, in the case of more numbers, so on): [30] is thus converted into [41], and we have the required equation

$$[41] = [3 \text{◇} 001),$$

where the *rationale* of this algorithmic process appears by the explanation, *ante*, No. 68.

71. As another example take

$$[8001] = [601 \text{◇} 003).$$

To [601], *quà* quartic transformation, belongs the number 4; and from 4, (003) we form 5, (114); where the 5 indicates that [601] is to be changed into [6010]; then (114), (003), writing them in the form $1^2 2^0 3^{-1} 4^1$, show that to the numbers of [6010] we are to add 2, 0, −1, 1; thus changing the symbol into [8001], so that we have the required relation.

72. Mr. Clifford calculated in this way the following table, showing how any transformation of an order not exceeding 8 can be expressed by means of a series of quadric transformations; the symbols Cr. 3, Cr. 4 . 1; 4 . 2, &c., refer to the order and number of Cremona's tables, *ante*, No. 41.

Cr. 3 . $[41] = [3 \text{◇} 001)$

Cr. 4 . 1 $= [601] = [41 \text{◇} 002) = [3 \text{◇} 001 \text{◇} 002)$

4 . 2 $= [330] = [3 \text{◇} 000)$
$= [41 \text{◇} 011) = [3 \text{◇} 001 \text{◇} 011)$

Cr. 5 . 1 $= [8001] = [601 \text{◇} 003) = [3 \text{◇} 001 \text{◇} 002 \text{◇} 003)$

5 . 2 $= [3310] = [41 \text{◇} 001) = [3 \text{◇} 001 \text{◇} 001)$

5 . 3 $= [0600] = [330 \text{◇} 111) = [3 \text{◇} 000 \text{◇} 111) = [3 \text{◇} 001 \text{◇} 011 \text{◇} 111)$

Cr. 6 . 1 $= [10, 0, 0, 0, 1] = [8001 \text{◇} 004) = [3 \text{◇} 001 \text{◇} 002 \text{◇} 003 \text{◇} 004)$

6 . 2 $= [14200] = [330 \text{◇} 011) = [3 \text{◇} 000 \text{◇} 011) = [3 \text{◇} 001 \text{◇} 011 \text{◇} 011)$

6 . 3 $= [41300] = [601 \text{◇} 011) = [3 \text{◇} 001 \text{◇} 002 \text{◇} 011)$
$= [3310 \text{◇} 022) = [3 \text{◇} 001 \text{◇} 001 \text{◇} 022)$ $\Big\} = [3 \text{◇} 001 \text{◇} 000)$

6 . 4 $= [34010] = [330 \text{◇} 002) = [3 \text{◇} 000 \text{◇} 002)$
$= [3310 \text{◇} 013) = [3 \text{◇} 001 \text{◇} 001 \text{◇} 013)$

Cr. 7.1 = [12, 00001] = [10, 0001 ⨟ 005) = [3 ⨟ 001 ⨯ 002 ⨯ 003 ⨯ 004 ⨯ 005)
7.2 = [330 ⨟ 001) = [3 ⨟ 000 ⨯ 001) = [232100]
7.3 = [034000] = [3310 ⨟ 111) = [3 ⨟ 001 ⨯ 001 ⨯ 111)
7.4 = [503100] = [601 ⨟ 001) = [3 ⨟ 001 ⨯ 002 ⨯ 001)
7.5 = [350010] = [3310 ⨟ 003) = [3 ⨟ 001 ⨯ 001 ⨯ 003)

Cr. 8.1 = [14, 000001] = [3 ⨟ 001 ⨯ 002 ⨯ 003 ⨯ 004 ⨟ 005 ⨯ 006)
8.2 = [3230100] = [3310 ⨟ 002) = [3 ⨟ 001 ⨯ 001 ⨯ 002)
8.3 = [1322000] = [3310 ⨟ 011) = [3 ⨟ 001 ⨯ 001 ⨯ 011)
8.4 = [0070000] = [034000 ⨟ 222) = [3 ⨟ 001 ⨯ 001 ⨯ 111 ⨯ 222)
8.5 = [3600010] = [34010 ⨟ 004) = [330 ⨟ 002 ⨯ 004)
= [3 ⨟ 000 ⨯ 002 ⨯ 004)
8.6 = [6013000] = [601 ⨟ 000) = [3 ⨟ 001 ⨯ 002 ⨯ 000)
8.7 = [0520100] = [0600 ⨟ 002) = [3 ⨟ 000 ⨯ 111 ⨯ 222)
8.8 = [2051000] = [41300 ⨟ 112) = [3 ⨟ 001 ⨯ 000 ⨯ 112)
8.9 = [3303000] = [3 ⨟ 000 ⨯ 000 ⨯ 000)

73. The reduction as above of a transformation to a series of quadric transformations, enables the determination of the reciprocal transformation; or, what is the same thing, the determination of the Jacobian of the first figure; see the example, *ante*, No. 67, where it appears that the reciprocal transformation of [41] is [41]. But I do not see any easy algorithmic process for the determination of the reciprocal transformation, or still less any general form in which the result can be expressed; and I do not at present pursue the inquiry.

The Rational Transformation between Two Spaces.

74. The general principles have been already explained: the two systems $x' : y' : z' : w' = X : Y : Z : W$ and $x : y : z : w = X' : Y' : Z' : W'$ must be derivable the one from the other; and starting with the first system, this will be the case if only the surfaces $X=0$, $Y=0$, $Z=0$, $W=0$ have a common intersection equivalent to n^3-1 points of intersection, but not equivalent to a complete common intersection of n^3 points. The last mentioned circumstance would arise, if the condition of the common intersection should impose upon the surface more than $\frac{1}{6}(n+1)(n+2)(n+3)-4$ conditions; viz., the surfaces would then be connected by an identical equation or syzygy $\alpha X+\beta Y+\gamma Z+\delta W = 0$. The common intersection is a figure composed of points and curves: say it is the principal system in the first space; the problem is, to determine a principal system equivalent to n^3-1 points of intersection, but such that the number of conditions to be satisfied by a surface passing through it is not more than $\frac{1}{6}(n+1)(n+2)(n+3)-4$.

75. The following locutions are convenient. We may say that the number of conditions imposed upon a surface of the order n which passes through the common intersection is the *Postulation* of this intersection; and that the number of points represented by the common intersection (in regard to the points of intersection of any three surfaces each of the order n which pass through it) is the *Equivalence* of this intersection. The conditions above referred to are thus

$$\text{Equivalence} = n^3 - 1,$$
$$\text{Postulation} \not> \tfrac{1}{6}(n+1)(n+2)(n+3) - 4.$$

76. It would appear by the analogy of the rational transformation between two planes, that the only cases to be considered are those for which $\text{Postulation} = \tfrac{1}{6}(n+1)(n+2)(n+3) - 4$; but I cannot say whether this is so.

77. In the transformation between two planes, the two conditions lead, as was seen, to the result that the curve $aX+bY+cZ=0$ is unicursal. I do not see that in the present case of two spaces, the two conditions lead to the corresponding result that the surface $aX+bY+cZ+dW=0$ is unicursal; that this is so, appears, however, at once from the general notion of the rational transformation. In fact, the equation in question $aX+bY+cZ+dW=0$ is satisfied by $x:y:z:w = X':Y':Z':W'$ and $ax'+by'+cz'+dw'=0$; the last equation determines the ratios $x':y':z':w'$ in terms of two arbitrary parameters (say these are $x':y'$ and $x':z'$), and we have then $x:y:z:w$ proportional to rational functions of these two parameters; that is, the surface $aX+bY+cZ+dW=0$ is unicursal. And similarly the surface $aX'+bY'+cZ'+dW'=0$ is unicursal.

78. In the most general point of view, the principal system will contain a given number of points which are simple points, a given number which are quadriconical points, a given number which are cubiconical points, &c. &c., on the surfaces; and similarly a given number of curves which are simple curves, a given number which are double curves, &c. &c., on the surfaces. But, to simplify, I will consider that it includes only points which are simple points, and a curve which is a simple curve on the surfaces: this curve may, however, break up into separate curves, and we thus, in fact, admit the case where there are any number of separate curves each of them a simple curve on the surfaces. It is right to remark that we cannot assert *à priori*—and it is not in fact the case—that the principal system in the second space will be subject to the like restrictions: starting with such a principal system in the first space, we may be led in the second space to a principal system including a curve which is a double curve on the surfaces; an instance of this will in fact occur.

79. It is shown (Salmon's Solid Geometry, 2nd ed., p. 283), that in the intersection of three surfaces of the orders μ, ν, ρ respectively, a curve of intersection of the order m and class r counts as $m(\mu+\nu+\rho-2)-r$ points of intersection. For a curve without actual double points or stationary points, we have $r = m(m-1)-2h$, where h is the number of apparent double points; or, substituting, we have the curve counting for $m(\mu+\nu+\rho-2)-m(m-1)+2h$ points of intersection; this is in fact a more general form of the formula, inasmuch as it extends to the case of a curve with actual double points and stationary points. Or, what is the same thing, the three surfaces intersecting in the curve of the order m with h apparent double points, will besides intersect in $\mu\nu\rho-m(\mu+\nu+\rho-2)+m(m-1)-2h$ points; viz., the curve may, besides the apparent double points, have actual double points and stationary points; but these do not affect the formula.

80. Some caution is necessary in the application of the theorem. For instance, to consider cases that will present themselves in the sequel: let the surfaces be cubics ($\mu=\nu=\rho=3$); the number of remaining intersections is given as $= 27-7m+m(m-1)-2h$. Suppose that the curve consists of four non-intersecting lines, $m=4$, $h=6$, the number is given as $=-1$. But observe in this case there are two lines each meeting the four given lines; that is, any cubic surface passing through the four given lines meets these two lines each of them in four points, that is, the cubic passes also through each of the two lines; the complete *curve*-intersection of the surfaces is made up of the six lines $m=6$, $h=7$ (since each of the two lines, as intersecting the four lines, gives actual double points, but the two lines together give one apparent double point), and the expression for the number of the remaining points of intersection becomes $= 27-42+30-14=1$, which is correct.

81. Similarly, if the given curve of intersection be a conic and two non-intersecting lines, there is here in the plane of the conic a line meeting each of the two given lines, and therefore meeting the cubic surface, in four points, that is, lying wholly in the cubic surface: the complete *curve*-intersection consists of the conic, the two given lines, and the last mentioned line $m=5$, $h=5$, and the number of points of intersection is $= 27-35+20-10=2$, which is correct. Again, if the given curve of intersection be two conics, here the line of intersection of the planes of the conics lies in the cubic surface; or, for the complete *curve*-intersection we have $m=5$, $h=4$; and the number of points is $27-35+20-8=4$. If in this last case each or either of the conics become a pair of intersecting lines, or if in the preceding case the conic becomes a pair of intersecting lines, the results remain unaltered.

82. If a surface of the order μ pass through a curve of the order m and class r without stationary points or actual double points, this imposes on the surface a number of conditions $=(\mu+1)\,m-\frac{1}{2}r$. In the case in question, the value of r is $=m\,(m-1)-2h$; or, substituting, the number of conditions is $=(\mu+1)\,m-\frac{1}{2}m\,(m-1)+h$; and the formula in this form holds good even in the case where the curve has stationary points and actual double points. Thus $\mu=3$, the number of conditions is $=4m-\frac{1}{2}m\,(m-1)+h$. If the curve be a line, $m=1$, $h=0$, number of conditions is $=4$; if the curve be a pair of non-intersecting lines, $m=2$, $h=1$, number of conditions is $=8$. And so in general, if the curve consist of k non-intersecting lines ($k=4$ at most), then $m=k$, $h=\frac{1}{2}k\,(k-1)$, and the number of conditions is $=4k$. If the curve be a conic, or a pair of intersecting lines, $m=2$, $h=1$, and the number of conditions is $=7$. If the curve consist of k lines, such that there are θ pairs of intersecting lines, then $m=k$, $h=\frac{1}{2}k\,(k-1)-\theta$, and the number of conditions is $=4k-\theta$. It is obvious that, the number of conditions for a line being $=4$, that for the k lines with θ intersecting pairs must have the foregoing value $4k-\theta$. In fact, when the lines do not intersect, we take on each line 4 points, and the cubic surface passing through any such 4 points will contain the line; but for two lines which intersect, taking this point, and on each of the intersecting lines 3 other points, the cubic surface through the 7 points will pass through the two lines; and so in other cases.

83. The formula must, in some instances, be applied with caution. Thus, given five non-intersecting lines $k=5$, $\theta=0$, and the number of conditions is $=20$; and a cubic surface cannot be, in general, made to pass through the lines. But if the five lines are met by any other line, then a cubic surface, if it pass through the five lines, will pass through this sixth line; for the six lines $k=6$, $\theta=5$, and the number of conditions is $24-5=19$; so that there is a determinate cubic surface through the six lines, and therefore through the five lines related in the manner just referred to.

84. Recurring to the problem of transformation, it appears by what precedes, that if the principal system in the first plane consists of α_1 points, and of a curve of the order m_1 with h_1 apparent double points (the α_1 points being simple points, and the curve a simple curve on the surfaces), then the conditions for a transformation are

$$(3n-2)\,m_1-\ m_1\,(m_1-1)+2h_1+\alpha_1=n^3-1,$$
$$(n+1)\,m_1-\tfrac{1}{2}m_1\,(m_1-1)+\ h_1+\alpha_1=\tfrac{1}{6}\,(n+1)\,(n+2)\,(n+3)-4,$$

where, in the second line, instead of $\not>$ I have written $=$. I remark, in passing, that I have ascertained that an actual triple point counts as an apparent double point; or, what is the same thing, that if the

curve has t_1 actual triple points, then we may, instead of h_1, write h_1+t_1. The equations give

$$m_1(4n-5-m_1) = \tfrac{1}{3}(n-1)(5n^2-n-12)-2h_1,$$
$$(n-4)m_1-a = \tfrac{1}{3}(n-1)(2n^2-4n-15),$$

to which may be joined

$$(3n+8)m_1-2m_1(m_1-1)+4h_1+5a_1 = (n-1)(6n+17).$$

The first two equations for the successive values of n give

$n=2$, $m_1(3-m_1) = 2-2h_1$,	$2m_1+a_1 = 5$;
$n=3$, $m_1(7-m_1) = 20-2h_1$,	$m_1+a_1 = 6$;
$n=4$, $m_1(11-m_1) = 64-2h_1$,	$a_1 = -1$;
$n=5$, $m_1(15-m_1) = 144-2h_1$,	$-m_1+a_1 = -20$;
$n=6$, $m_1(19-m_1) = 270-2h_1$,	$-2m_1+a_1 = -55$;
&c. &c.	&c.

85. It is remarkable that for $n=4$ there is no solution of the geometrical problem; in fact, $a_1=-1$, a negative value of a_1, shows that this is so. For the higher values of n, there seem to be solutions with large values of m_1, h_1, a_1; for example, $n=5$, we have $m_1=20+a_1$, is $=20$ at least. Writing $m_1=20$, we have $-100=144-2h_1$, or $2h_1=244$. The highest value of $2h_1$ is $=(m_1-1)(m_1-2)$, which for $m_1=20$ is $=342$; or the foregoing value $2h_1=244$ is admissible. Thus $m_1=20$, $h_1=122$, $a_1=0$ gives a solution; and, moreover, any larger value of m_1, say $m_1=20+a$, gives an admissible solution, $m_1=20+a$, $h_1=122+\frac{1}{2}a(a+25)$, $a_1=a$. And so for $n=6$, &c.; but I have not further examined any of these cases, and do not understand them.

There remain the cases $n=2$, $n=3$. For $n=2$, since $2m_1+a_1=5$, we have $m_1=0$, 1, or 2; $m_1=0$ gives $h_1=0$, which is not admissible. The remaining solutions are $m_1=1$, $h_1=0$, $a_1=3$; and $m_1=2$, $h_1=0$, $a_1=1$.

For $n=3$, since $m_1+a_1=6$, we have $m_1=0$, 1, 2, 3, 4, 5, or 6. $m_1=0$ gives $h_1=10$; $m_1=1$ gives $h_1=7$; $m_1=2$ gives $h_1=5$; $m_1=3$ gives $h_1=4$: these values are not geometrically admissible. The remaining cases are $m_1=4$, $h_1=4$, $a_1=2$; $m_1=5$, $h_1=5$, $a_1=1$; $m_1=6$, $h_1=7$, $a_1=0$.

86. The reciprocal transformation is in every case of the order $n'=n^2-m_1$. Hence considering the quadric transformations—

First, the case $n=2$, $m_1=1$, $h_1=0$, $a_1=3$; the reciprocal transformation is of the order $n'=3$. Suppose for a moment that the principal system in the second space is of the same nature as that above considered in the first space, consisting of a_1' points, and a curve of the order m_1' with h_1' apparent double points (the a_1' points each a simple point, and the curve a simple curve on the surfaces $X'=0$, &c.).

Passing back to the original transformation, we should have $2 = 9 - m'_1$, that is, $m'_1 = 7$. But it has just been seen that, for $n=3$, the only values of m_1 are 4, 5, 6; hence for $n' = 3$ we cannot have $m'_1 = 7$. The explanation is, that the principal system in the second space *is not of the form in question;* it, in fact, consists (as will appear) of three lines each a simple line, and of another line which is a double line on the surfaces $X' = 0$, &c. In the intersection of any two of these surfaces, the three lines count each once, the double line four times, and the order of the curve of intersection is thus $3+4 = 7$, as it should be. The principal system may be characterized $\alpha'_1 = 0$, $m'_1 = 3$, $h'_1 = 3$, $m'_2 = 1$, $h'_2 = 0$.

Next, the case $n=2$, $m_1 = 2$, $h_1 = 0$, $\alpha_1 = 1$: the reciprocal transformation is of the order $n' = 2$; it is evidently not of the form above considered (for this would make the original transformation to be of the order 3). Hence, assuming (as it seems allowable to do) that the principal system does not contain any multiple point or curve, the reciprocal transformation will be of the same form as the original one; viz., we shall have $n' = 2$, $m'_1 = 2$, $h'_1 = 0$, $\alpha'_1 = 1$.

87. Considering next the cubic transformations, or those belonging to $n=3$; in the case $m_1 = 4$, $h_1 = 4$, $\alpha_1 = 2$, the reciprocal transformation is of the order $9 - 4 = 5$; and in the case $m_1 = 5$, $h_1 = 5$, $\alpha_1 = 1$, the reciprocal transformation is of the order $9 - 5 = 4$: I do not consider these cases. But $m_1 = 6$, $h_1 = 7$, $\alpha_1 = 0$, the reciprocal transformation is of the order $9 - 6 = 3$; and assuming (as seems allowable) that the principal system does not contain any multiple point or curve, it must be of the same form as the original transformation, that is, we must have $n' = 3$, $m'_1 = 6$, $h'_1 = 7$, $\alpha'_1 = 0$.

88. The transformations to be studied are thus,—1° The quadri-quadric transformation $n=2$, $m_1 = 2$, $h_1 = 0$, $\alpha_1 = 1$, and $n' = 2$, $m'_1 = 2$, $h'_1 = 0$, $\alpha'_1 = 1$; the principal system in each space consists of a point and of a conic (which may be a pair of intersecting lines); and the surfaces are quadrics. 2° The quadri-cubic transformation $n = 2$, $m_1 = 1$, $h_1 = 0$, $\alpha_1 = 3$, and $n' = 3$, $\alpha'_1 = 0$, $m'_1 = 3$, $h'_1 = 3$, $m'_2 = 1$, $h'_2 = 0$: in the first space the principal system consists of three points and a line, and the surfaces are quadrics: in the second figure, the principal system consists of three simple lines and a double line; and the surfaces are cubic surfaces passing through this principal system, that is, they are cubic scrolls. 3° The cubo-cubic transformation $n=3$, $\alpha_1 = 0$, $m_1 = 6$, $h_1 = 7$, and $n' = 3$, $\alpha'_1 = 0$, $m'_1 = 6$, $h'_1 = 7$; in each space the principal system is a sextic curve with seven apparent double points (but there are different cases to be considered according as the sextic curve does or does not break up into inferior curves), and the surfaces are cubic surfaces through the sextic curve.

The Quadri-quadric Transformation between Two Spaces.

89. Starting from the equations $x' : y' : z' : w' = X : Y : Z : W$, we have here $X=0$, &c., quadric surfaces passing through a given point and a given conic (which may be a pair of intersecting lines). Take $x=0$, $y=0$, $z=0$ for the coordinates of the given point; $w=0$ for the equation of the plane of the conic; the conic is then given as the intersection of this plane by a cone having the given point for its vertex; or say the equations of the conic are $w=0$, $(a, \ldots \mathcal{X}x, y, z)^2 = 0$; the general equation of a quadric through the point and conic is $w(\alpha x+\beta y+\gamma z)+\delta(a, \ldots \mathcal{X}x, y, z)^2 = 0$; and it hence appears that the equations of the transformation may be taken to be

$$x' : y' : z' : w' = xw : yw : zw : (a, \ldots \mathcal{X}x, y, z)^2;$$

these give at once a reciprocal system of the same form; viz., the two sets are $x' : y' : z' : w' = xw : yw : zw : (a, \ldots \mathcal{X}x, y, z)^2$,

and $x : y : z : w = x'w' : y'w' : z'w' : (a, \ldots \mathcal{X}x', y', z')^2$.

90. The Jacobian of the first space is at once found to be

$$w^2(a, \ldots \mathcal{X}x, y, z)^2 = 0;$$

that of the second space is of course

$$w'^2(a, \ldots \mathcal{X}x', y', z')^2 = 0.$$

The two spaces are similar to each other; we may say that there is in each of them a principal point and a principal conic; that the plane of the conic is the principal plane, and the cone having its vertex at the point and passing through the conic is the principal cone. To the principal point of either space corresponds any point whatever in the principal plane of the other space; and conversely. More definitely, the points of the one principal plane and the infinitesimal elements of direction through the principal point of the other space correspond according to the equation $x : y : z = x' : y' : z'$. To any point on the principal conic of either space corresponds in the other space, not a mere element of direction through the principal point of the other space, but a line of the principal cone; that is, to the points of the principal conic of the one space correspond the lines of the principal cone of the other space. The Jacobian of either space, consisting of the principal plane twice, and of the principal cone, is thus the principal counter-system of the other space.

91. [Writing $(a, \ldots \mathcal{X}x, y, z)^2 = x^2+y^2+z^2$, $w=w'=1$, the equations of transformation become

$$x' : y' : z' : 1 = x : y : z : x^2+y^2+z^2,$$

and $$x : y : z : 1 = x' : y' : z' : x'^2+y'^2+z'^2,$$

or, what is the same thing, if for shortness

$$x^2+y^2+z^2=r^2, \quad x'^2+y'^2+z'^2=r'^2,$$

the equations are

$$x'=\frac{x}{r^2}, \quad y'=\frac{y}{r^2}, \quad z'=\frac{z}{r^2}; \quad \text{and} \quad x=\frac{x'}{r'^2}, \quad y=\frac{y'}{r'^2}, \quad z=\frac{z'}{r'^2},$$

whence also $rr'=1$; this is the well known transformation by reciprocal radius vectors.]

92. The principal conic may be a pair of intersecting lines; taking its equations to be $w=0$, $xy=0$, the equations of transformation here become $x' : y' : z' : w' = xw : yw : zw : xy$,
and $x : y : z : w = x'w' : y'w' : z'w' : x'y'$.
There is no difficulty in the further development of the theory.

The Quadri-cubic Transformation between Two Spaces.

93. It will be convenient to have the unaccented letters (x, y, z, w) referring to the cubic surfaces. I will therefore take the quadric surfaces in the second figure; viz., I will start from the equations $x : y : z : w = X' : Y' : Z' : W'$, where $X'=0$, $Y'=0$, $Z'=0$, $W'=0$ are quadric surfaces passing through three fixed points (say the principal points) and through a fixed line (say the principal line) in the second figure. Taking $x'=0$, $y'=0$ for the planes passing through the principal line and through two of the principal points respectively; $z'=0$ for the plane passing through the three principal points, $w'=0$ for an arbitrary plane passing through the first mentioned two principal points, the implicit factors of x', y', w' may be so determined that for the third principal point $x'=y'=-w'$. That is, we shall have

for principal line $\quad x'=0, \ y'=0,$
for principal points $\quad (x'=0, \ z'=0, \ w'=0),$
„ $\quad (y'=0, \ z'=0, \ w'=0),$
„ $\quad (x'=y'=-w', \ z'=0),$

and this being so, the equation of a quadric surface through the principal points and line will be

$$(\alpha x'+\beta y')z'+\gamma x'(y'+w')+\delta y'(x'+w'),$$

and the equations of transformation may be taken to be

$$x : y : z : w = x'z' : y'z' : x'(y'+w') : y'(x'+w').$$

94. Writing these in the extended form

$$x : y : z : w : x-y : z-w =$$
$$x'z' : y'z' : x'(y'+w') : z'(x'+w') : z'(x'-y') : w'(x'-y')$$

and forming also the equation

$$xy : (xw-yz) = z' : x'-y',$$

we at once derive the reciprocal system of equations

$$x' : y' : z' : w' = x(xw-yz) : y(xw-yz) : (x-y)xy : (z-w)xy,$$

so that this is a cubic transformation. And the cubic surface in the first space (corresponding to an arbitrary plane $ax'+by'+cz'+dw'=0$ of the second space) is $(ax+by)(xw-yz)+c(x-y)xy+d(z-w)xy=0$; viz., this is a cubic surface having the fixed double line $(x=0, y=0)$, the fixed simple lines $(x=0, z=0)$, $(y=0, w=0)$, and $(x-y=0, z-w=0)$; it has also the variable simple line $(dz+cx=0, dw+cy=0)$. The principal figure of the first space thus consists of the three simple lines $(x=0, z=0)$, $(y=0, w=0)$, $(x-y=0, z-w=0)$, and of the line $(x=0, y=0)$, a double line counting four times in the intersection of two of the cubic surfaces.

95. The cubic surface as having the double line $(x=0, y=0)$ is a cubic scroll, and this line is the nodal directrix thereof; the line $(dz+cx=0, dw+cy=0)$ is the simple directrix; the lines $(x=0, z=0)$, $(y=0, w=0)$, $(x-y=0, z-w=0)$ are at once seen to be lines meeting each of these directrix lines; and they are generating lines of the scroll. To explain the generation of the scroll, observe that the section by any plane is a cubic curve having a given double point (viz., the intersection of the plane with the nodal directrix); and three other given points (viz., the intersections of the plane with the three generating lines respectively); this cubic also passes through the intersection of the plane with the simple directrix. Conversely, if the plane be assumed at pleasure, and if, taking for the simple directrix any line which meets the given generating lines, we draw a cubic as above, then the scroll is the scroll generated by a line which meets each of the directrix lines, and also the cubic.

If the plane be taken to pass through any generating line, then the cubic section breaks up into this line, and a conic; the conic does not meet the simple directrix, but it meets the nodal directrix; and any such conic will serve as a directrix; viz., the scroll is generated by the lines which meet the two directrix lines and the conic.

96. Any two scrolls as above meet in the three fixed generating lines, and in the nodal directrix counting four times; they consequently meet besides in a curve of the second order, which is a conic (one of the conics just referred to). In order to further explain the theory, suppose for a moment that the two scrolls had only a common nodal directrix; they would besides meet in a quintic curve; this curve would meet the nodal directrix in four points, viz., the points at which the two scrolls have a common tangent plane. Now if at any point of the nodal directrix the two scrolls have a common generating line, then the plane through this line and the nodal line is one of the two tangent

planes of each scroll; that is, the scrolls have this plane for a common tangent plane. Hence, in the case of the common three generating lines, the points where these meet the nodal line are three of the four points just referred to; there remains therefore one point, which is the point where the conic meets the nodal line; through this point there are for each of the scrolls two generating lines; one of these for the first scroll, and one for the second scroll, lie in a plane with the nodal line; the other two determine the plane of the conic; and the tangent to the conic at its intersection with the nodal line is the intersection of the plane of the conic with the plane of the first-mentioned two generating lines.

97. Analytically we have the two equations

$$c\,(x-y)\,xy+(ax+by)\,(xw-yz)+d\,(z-w)\,xy=0,$$
$$c'\,(x-y)\,xy+(a'x+b'y)\,(xw-yz)+d'\,(z-w)\,xy=0;$$

or, combining these equations so as to eliminate successively the terms in $x\,(xw-yz)$ and $y\,(xw-yz)$, and for this purpose writing

$$(bc'-b'c,\ ca'-c'a,\ ab'-a'b,\ ad'-a'd,\ bd'-b'd,\ cd'-c'd)=(\mathrm{a},\mathrm{b},\mathrm{c},\mathrm{f},\mathrm{g},\mathrm{h}),$$

and therefore

$$\mathrm{af}+\mathrm{bg}+\mathrm{ch}=0,$$

we have

$$\mathrm{b}\,(x-y)\,x-\mathrm{c}\,(xw-yz)-\mathrm{f}\,(z-w)\,x=0,$$
$$-\mathrm{a}\,(x-y)\,y+\mathrm{c}\,(xw-yz)-\mathrm{g}\,(z-w)\,y=0,$$

and multiplying the first of these by $\mathrm{c}+\mathrm{g}$ and the second by $\mathrm{c}-\mathrm{f}$, and adding, the whole divides by $x-y$, and the final result is

$$(\mathrm{c}+\mathrm{g})\,(\mathrm{b}x-\mathrm{f}z)-(\mathrm{c}-\mathrm{f})\,(\mathrm{a}y+\mathrm{g}w)=0;$$

viz., this is the equation of the plane of the conic.

98. Any two scrolls as above meeting in a conic, a third scroll will meet the conic in six points; but these include the point on the nodal directrix twice, and the points on the three fixed generating lines each once; there is left a single point of intersection, viz., this is the *one variable* point of intersection of the three scrolls; which is in accordance with the theory.

99. For the Jacobian of the second space, we have

$$\begin{vmatrix} z', & 0, & x', & 0 \\ 0, & z', & y', & 0 \\ y'+w', & x', & 0, & 0 \\ y', & x'+w', & 0, & y' \end{vmatrix}=0;$$

that is, $2x'y'z'(x'-y')=0$; viz., $z'=0$ is the plane containing the three principal points; and $x'=0$, $y'=0$, $x'-y'=0$ are the planes which pass through the principal line and the three principal points respectively.

100. For the Jacobian of the first space, we have

$$\begin{vmatrix} 2xw-yz, & -xz, & -xy, & x^2 \\ yw, & xw-2yz, & -y^2, & xy \\ 2xy-y^2, & x^2-2xy, & 0, & 0 \\ (z-w)y, & (z-w)x, & xy, & -xy \end{vmatrix} = 0;$$

that is, $3x^2y^2(x-y)^2(xw-yz) = 0$; viz., $x=0$, $y=0$, $x-y=0$ are the planes through the nodal directrix and the three fixed generators respectively (each plane therefore occurring twice); and $xw-yz=0$ is the quadric scroll generated by the lines which meet each of the three generators $(x=0, z=0)$, $(y=0, w=0)$, $(x-y=0, z-w=0)$; this scroll passing also through the nodal directrix $x=0, y=0$.

The Cubo-cubic Transformation between Two Spaces.

101. The principal system in the first space is a sextic curve with 7 apparent double points; but this curve may be either a single curve, or it may break up into inferior curves. I have not examined all the cases which may arise; but the two extreme cases are—(A) The sextic curve breaks up into six lines, viz., two non-intersecting lines, and four other lines each meeting each of the two lines (this implies that no two of the four lines meet each other): here the two lines give 1 apparent double point, and the four lines give 6 apparent double points; total number is $= 7$, as it should be. (B) The curve is a proper sextic curve, with 7 apparent double points: this gives, as will be shown, the general lineo-linear transformation. The two cases are each of them symmetrical.

(A) *The Principal System consists of Six Lines.*

102. Taking in the first space, for the equations of the two lines, $(x=0, y=0)$ and $(z=0, w=0)$, and for the equations of the four lines, $(x=0, z=0)$, $(y=0, w=0)$, $(x-y=0, z-w=0)$, $(x-py=0, z-qw=0)$, then, if the equations of transformation are taken to be

$$\begin{aligned} x'-py' : x'-y' : z'-qw' : z'-w' = \quad & (x-py)(xw-yz) \\ & : (x-y)(qxw-pyz) \\ & : (z-qw)(xw-yz) \\ & : (z-w)(qxw-pyz); \end{aligned}$$

these lead conversely (see *post*, No. 104) to a like system,

$$\begin{aligned} x-py : x-y : z-qw : z-w = \quad & (x'-py')\,\mathrm{M}' \\ & : (x'-y')\,\mathrm{N}' \\ & : (z'-qw')\,\mathrm{M}' \\ & : (z'-w')\,\mathrm{N}', \end{aligned}$$

where for shortness

$$\begin{aligned} M' &= p(q-1)^2 x'w' - q(p-1)^2 y'z' + (pq-1)(p-q) y'w', \\ N' &= (q-1)^2 x'w' - (p-1)^2 y'z' + (pq-1)(p-q) y'w'; \end{aligned}$$

or, as these are better written,

$$\begin{aligned} M' &= -q(p-1) y' (\overline{p-1}\, z' - \overline{pq-1}\, w') + p(q-1) w' (\overline{q-1}\, x' - \overline{pq-1}\, y'), \\ N' &= -(p-1) y' (\overline{p-1}\, z' - \overline{pq-1}\, w') + (q-1) w' (\overline{q-1}\, x' - \overline{pq-1}\, y'). \end{aligned}$$

Hence the principal system in the second plane is composed of the two non-intersecting lines $(x'=0,\ y'=0)$, $(z'=0,\ w'=0)$ and the four lines $(\overline{p-1}\, z' - \overline{pq-1}\, w' = 0,\ \overline{q-1}\, x' - \overline{pq-1}\, y' = 0)$, $(y'=0, w'=0)$, $(x'-y'=0,\ z'-w'=0)$, $(x'-py'=0,\ z'-qw'=0)$, each meeting each of the two lines.

103. The Jacobian of the first space is

$$\begin{vmatrix} 2xw-yz-pyw, & -xz-pxw+2pyz, & -y(x-py), & x(x-py) \\ 2qxw-pyz-qyw, & -pxz-qxw+2pyz, & -py(x-y), & qx(x-y) \\ w(z-qw), & -z(z-qw), & xw-2yz+qyw, & xz-2qxw+qyz \\ qw(z-w), & -pz(z-w), & qxw-2pyz+pyw, & qxz-2qxw+pyz \end{vmatrix} = 0,$$

viz., this is $xyzw(x-y)(x-py)(z-w)(z-qw) = 0$, the equation of the planes each passing through one of the four lines and one of the two lines.

Similarly, the Jacobian of the second space is

$$(\overline{q-1}\, x' - \overline{pq-1}\, y')(\overline{p-1}\, z' - \overline{pq-1}\, w')(x'-y')(z'-w') \times (x'-py')(z'-qw') y'w' = 0;$$

viz., this is the equation of the eight planes each passing through one of the four lines and one of the two lines.

104. To effect the foregoing transformation, writing

$$\begin{aligned} x' : y' : z' : w' = {} & (x-py)(xw-yz) \\ & : (x-y)(qxw-pyz) \\ & : (z-qw)(xw-yz) \\ & : (z-w)(qxw-pyz); \end{aligned}$$

or what will ultimately be the same thing, but it is more convenient for working with,

$$\begin{aligned} x' &= (x-py)(xw-yz), \\ y' &= (x-y)(qxw-pyz), \\ z' &= (z-qw)(xw-yz), \\ w' &= (z-w)(qxw-pyz); \end{aligned}$$

these give

$$\begin{aligned} x-py &= M'x', \\ x-y &= N'y', \\ z-qw &= M'z', \\ z-w &= N'w', \end{aligned}$$

where M′, N′ are quantities which have to be determined; and thence

$$\begin{aligned}(1-p)\,x &= \mathrm{M}'x' - p\mathrm{N}'y',\\ (1-p)\,y &= \mathrm{M}'x' - \mathrm{N}'y',\\ (1-q)\,z &= \mathrm{M}'z' - q\mathrm{N}'w',\\ (1-q)\,w &= \mathrm{M}'z' - \mathrm{N}'w';\end{aligned}$$

whence also

$$(1-p)(1-q)(xw-yz) = \mathrm{N}'\{(q-1)\,x'w' - (p-1)\,y'z'\}\mathrm{M}' + pq\mathrm{N}'^2\{y'w' - z'^2\},$$

$$(1-p)(1-q)(qxw-pyz) =$$

$$\mathrm{M}'\{-(p-q)\,x'z'\mathrm{M}' + (pq-q)\,x'w' - pq+p\,y'z'\}\mathrm{N}';$$

but we have

$$\frac{xw-yz}{qxw-pyz} = \frac{x'}{y'} \div \frac{x-py}{w-y} = \frac{x'}{y'} \div \frac{\mathrm{M}'x'}{\mathrm{N}'y'} = \frac{\mathrm{N}'}{\mathrm{M}'};$$

or substituting

$$\begin{aligned}&\mathrm{M}'(\sqrt{q-1}\,x'w' - \sqrt{p-1}\,y'z') + \mathrm{N}'(p-q)\,y'w'\\ &= \mathrm{M}'(-\sqrt{p-q}\,x'z') + \mathrm{N}'(\sqrt{pq-q}\,x'w' - \sqrt{pq-p}\,y'z');\end{aligned}$$

that is

$$\mathrm{M}'\{\sqrt{q-1}\,x'w' - \sqrt{p-1}\,y'z' + \sqrt{p-q}\,x'z'\}$$

$$= \mathrm{N}'\{\sqrt{pq-q}\,x'w' - \sqrt{pq-p}\,y'z' - \sqrt{p-q}\,y'w'\};$$

or, what is the same thing,

$$\begin{aligned}\mathrm{M}' &= \sqrt{pq-q}\,x'w' - \sqrt{pq-p}\,y'z' - \sqrt{p-q}\,y'w',\\ \mathrm{N}' &= \sqrt{q-1}\,x'w' - \sqrt{p-1}\,y'z' + \sqrt{p-q}\,x'z';\end{aligned}$$

viz., M′, N′ having these values, the original equations

$$\begin{aligned}x' : y' : z' : w' = \;&(x-py)\,(xw-yz)\\ :\;&(x-y)\,(pxw-qyz)\\ :\;&(z-qw)\,(xw-yz)\\ :\;&(z-w)\,(pxw-qyz),\end{aligned}$$

give

$$x-py : x-y : z-pw : z-w = \mathrm{M}'x' : \mathrm{N}'y' : \mathrm{M}'z' : \mathrm{N}'w'.$$

If, in these equations, in place of (x', y', z', w') we write $(x-py, x-y, z-qw, z-w)$, the new values of M′, N′ are found to be

$$\begin{aligned}\mathrm{M}' &= p(q-1)\,x'w' - q(p-1)^2\,y'z' + (pq-1)(p-q)\,y'w',\\ \mathrm{N}' &= (q-1)^2\,x'w' - (p-1)^2\,y'z' + (pq-1)(p-q)\,y'w',\end{aligned}$$

and we have the formulæ of No. 102.

(B) *The Principal System of a Proper Sextic Curve; the Linee-linear Transformation between Two Spaces.*

105. I start with the linee-linear transformation, and show that this is in fact a transformation such that the principal system in either space is a sextic curve with seven apparent double points. I do not

attempt any formal proof, but assume that the linco-linear transformation is the most general one which gives rise to such a principal system.

We have between (x, y, z, w), (x', y', z', w') three linco-linear equations; writing these first under the form

$$(P_1, Q_1, R_1, S_1)(x', y', z', w') = 0,$$
$$(P_2, Q_2, R_2, S_2)(x', y', z', w') = 0,$$
$$(P_3, Q_3, R_3, S_3)(x', y', z', w') = 0,$$

we have $x' : y' : z' : w' = X : Y : Z : W$, where X, Y, Z, W are the determinants (each with its proper sign) formed out of the matrix

$$\begin{vmatrix} P_1, & Q_1, & R_1, & S_1 \\ P_2, & Q_2, & R_2, & S_2 \\ P_3, & Q_3, & R_3, & S_3 \end{vmatrix}.$$

106. Each of the surfaces $X=0$, $Y=0$, $Z=0$, $W=0$, or generally any surface $aX+bY+cZ+dW=0$, is thus a cubic surface passing through the curve

$$\left\| \begin{matrix} P_1, & Q_1, & R_1, & S_1 \\ P_2, & Q_2, & R_2, & S_2 \\ P_3, & Q_3, & R_3, & S_3 \end{matrix} \right\| = 0,$$

which is at once seen to be of the order 6. In fact any two of these surfaces, for instance

$$\begin{vmatrix} P_1, & Q_1, & R_1 \\ P_2, & Q_2, & R_2 \\ P_3, & Q_3, & R_3 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} P_1, & Q_1, & S_1 \\ P_2, & Q_2, & S_2 \\ P_3, & Q_3, & S_3 \end{vmatrix} = 0,$$

have in common a curve

$$\left\| \begin{matrix} P_1, & P_2, & P_3 \\ Q_1, & Q_2, & Q_3 \end{matrix} \right\| = 0,$$

which is of the order 3; they consequently besides intersect in a curve of the order 6, which is the before mentioned curve of intersection of all the surfaces. And it further appears that the number of the apparent double points is $= 7$; in fact the formula in the case of two surfaces of the orders μ, ν, the complete intersection of which consists of a curve of the order m with h apparent double points, and of a curve of the order m' with h' apparent double points, the numbers m, m', h, h' are connected by the equation $2(h-h') = (m-m')(\mu-1)(\nu-1)$. (Salmon's Solid Geom., 2nd Ed., p. 273.) Hence, in the case of the two cubic surfaces intersecting as above (since for the cubic curve we have $m'=3$, $h'=1$, and for the sextic $m=6$), the formula becomes $2(h-1) = 12$, that is $h=1+6=7$; or the number of apparent double points is $= 7$.

107. It thus appears that the principal system in the first plane is a curve of the order 6, with seven apparent double points: it is to be added that there are not in general any actual double points or sta-

tionary points, so that the class of the curve is $6.5-2.7, = 16$, and its deficiency is $\frac{1}{2}5.4-7, = 3$. For convenience I will refer to this as the curve Σ.

The transformation is obviously a symmetrical one; hence the principal system in the second space is in like manner a curve of the order 6, with seven apparent double points; say it is the curve Σ'.

108. Consider in the first space any point P on the curve Σ; for this point the three equations

$$\begin{aligned}(P_1, Q_1, R_1, S_1 \mathbin{\text{\textbigcircle}} x', y', z', w') &= 0,\\ (P_2, Q_2, R_2, S_2 \mathbin{\text{\textbigcircle}} \quad ,, \quad) &= 0,\\ (P_3, Q_3, R_3, S_3 \mathbin{\text{\textbigcircle}} \quad ,, \quad) &= 0,\end{aligned}$$

are not independent, but are equivalent to two linear equations in (x', y', z', w'); that is, to the point P on the curve Σ there corresponds in the second space, not a determinate point P′, but any point whatever on a certain line L′; or say to the point P on Σ there corresponds a line L′; and as P describes the curve Σ, L′ describes a scroll Π'; that is, to the curve Σ there corresponds a scroll Π', the principal counter-system in the second space. Similarly to the curve Σ' there corresponds a scroll Π, the principal counter-system of the first space.

109. The scroll Π is the Jacobian of the first space; and as such it is of the order 8, having the curve Σ for a triple line—and it thus appears that the Jacobian of the first space is a scroll (a theorem the analytical verification of which seems by no means easy). But without assuming the identity of the scroll Π with this Jacobian, or taking the order of the scroll to be known, I proceed to show that the scroll Π is the scroll generated by the lines each of which meets the curve Σ three times; it will thereby appear that the order is $= 8$, and that the curve is a triple line on the scroll.

Consider a point P′ on Σ', and the corresponding line L of the first space: take Θ' a plane in the second place; corresponding to it the cubic surface Θ in the first space. By imposing a single relation on the coefficients (a, b, c, d) in the equation $ax'+by'+cz'+dw' = 0$ of the plane Θ', we make it pass through the point P′; therefore by imposing this same *single* relation on the coefficients (a, b, c, d) of the cubic surface Θ, we make it pass through the line L; Θ is a cubic surface through Σ; and it is easy to see that the effect will be as above only if the line L cuts the curve Σ three times; this being so, the general cubic surface Θ meets L in three points (viz., the three intersections of L with Σ), and if Θ be made to pass through a fourth point on the line L, it will pass through the line L; it thus appears that the line L meets Σ three times, and consequently that the scroll Π is generated by the lines which meet Σ three times.

110. The theory of a scroll so generated is considered in my "Memoir on Skew surfaces, otherwise Scrolls."* Writing $m=6$, $h=7$ and therefore $\mathrm{M}[=-\frac{1}{2}m(m-1)+h]=-8$, the order of the scroll is $(\frac{1}{3}[m]^3+(m-2)\,\mathrm{M}=40-32)=8$; but calculating the values of

$$\mathrm{NG}(m^3)=\tfrac{1}{2}[m]^4+6m+\mathrm{M}(3[m]^2-12m+33)+\mathrm{M}^2.3,$$

$$\mathrm{NR}(m^3)=\tfrac{1}{18}[m]^6+\tfrac{3}{8}[m]^5-\tfrac{1}{2}[m]^3-3m$$
$$+\mathrm{M}(\tfrac{1}{3}[m]^4-\tfrac{1}{6}[m]^3-\tfrac{5}{2}m^2+8m-20)+\mathrm{M}^2(\tfrac{1}{2}[m]^2-2m);$$

these are found to be respectively $=0$; viz., there are no nodal generators, and no nodal residue; the sextic curve Σ is a triple curve on the surface, and there is not any other multiple line.

111. It may be remarked that any plane Θ' meets the sextic curve Σ' in six points; hence the corresponding cubic surface Θ contains six lines, generatrices of Π, and, therefore, each meeting the curve Σ three times; say six lines L. Through one of these lines L, draw to the cubic surface Θ a triple tangent plane meeting it in the line L and in two other lines, say M, N; this plane must meet Σ in three new points which must lie on the lines M, N; viz., one of these lines must pass through two of the points, and the other line through the third point.

Addition—September, 1870.

The formulæ of No. 84 are included in the following more general formulæ; viz., if the principal system consist of α_1 points, each a simple point, α_2 points each a quadri-conical point, α_3 points each a cubi-conical point, &c., and of a simple curve order m_1 with h_1 apparent double points, a double curve order m_2 with h_2 apparent double points, and so on; and if moreover, the curves m_1, m_2 intersect in $k_{1,2}$ points, the curves m_1, m_3 in $k_{1,3}$ points, &c.; then writing in general $\rho=\frac{1}{2}m(m-1)-h$; that is, $\rho_1=\frac{1}{2}m_1(m_1-1)-h_1$, $\rho_2=\frac{1}{2}m_2(m_2-1)-h_2$, &c., I find that the general condition of equivalence is

$$\left.\begin{aligned}
&\alpha_1+(3n-1)\,m_1 \quad -2\rho_1\\
+\;&8\alpha_2+(12n-16)\,m_2-16\rho_2\\
&\vdots\\
+\;&r^3\alpha_r+(3r^2n-2r^3)\,m_r-2r^3\rho_r\\
-\;&5k_{1,2}-8k_{1,3}\ldots-(3r-1)\,k_{1,r}\\
&\quad -28k_{2,3}\\
&\qquad\qquad \ldots-s^2(3r-s)\,k_{s,r}\,(s<r)
\end{aligned}\right\}=n^3-1;$$

and that the general condition of postulation is

* Phil. Trans., vol. cliii. 1863, pp. 453—483. See the Table S(m^3) &c., p.457; in the value of NR (m^3) instead of term $+3m$ read $-3m$.

$$\left.\begin{array}{l} a_1+\ (n+1)\,m_1-\ \rho_1 \\ +4a_2+(3n+1)\,m_2-5\rho_2 \\ \quad\vdots \\ +\frac{1}{6}r\,(r+1)\,(r+2)\,a_r \\ \quad +[\frac{1}{2}r\,(r+1)\,n-\frac{1}{6}r\,(r+1)\,(2r-5)]\,m_r \\ \quad\quad -\frac{1}{24}[(r-1)\,(r-2)\,(r-3)\,(r-4) \\ \quad\quad\quad +4r\,(r+1)\,(2r+1)]\,\rho_r \\ -2k_{1,2}-3k_{1,3}\ \ldots\ -rk_{1,r} \\ \quad -8k_{2,3} \\ \quad\quad\vdots \\ \quad -\frac{1}{2}s\,(s+1)(r+1-\frac{1}{3}s+2)\,k_{s,r}\,(s<r) \end{array}\right\} = \frac{1}{6}(n+1)(n+2)\,(n+3)-4:$$

in which formulæ it is however assumed that the curves have not any actual multiple points. This implies that if any one of the curves, say m_r, break up into two or more curves, the component curves do not intersect each other; for, of course, any such point of intersection would be an actual double point on the curve m_r. I believe, however, that the formulæ will extend to this case by admitting for s the value $s=r$; viz., if we suppose the curve m_r to be the aggregate of the two curves m_r', m_r'' intersecting in K_r points, then that the corresponding terms in the equivalence-equation are

$$(3r^2n-2r^3)(m_r'+m_r'')-2r^3(\rho_r'+\rho_r'')-2r^3\mathrm{K}_r,$$

and that those in the postulation-equation are

$$\begin{array}{l} [\frac{1}{2}r(r+1)n-\frac{1}{6}r\,(r+1)(2r-5)]\,(m_r'+m_r'') \\ \quad -\frac{1}{24}\,[(r-1)(r-2)(r-3)(r-4)+4r(r+1)(2r+1)]\,(\rho_r'+\rho_r'') \\ \quad -\frac{1}{6}r\,(r+1)(2r+1)\,\mathrm{K}_r. \end{array}$$

Let the r-tuple curve consist of three right lines meeting in a point: this is an actual triple point, and the formulæ do not apply. But calculating the postulation-terms by the formula, we have $m_r=3$, $\rho_r=\frac{1}{2}3.2-0,\ =3$; and the terms are

$$\begin{array}{l} [\frac{1}{2}r\,(r+1)\,n-\frac{1}{6}r\,(r+1)(2r-5)]\,3 \\ \quad\quad -\frac{1}{8}\,[(r-1)(r-2)(r-3)(r-4)+4r\,(r+1)(2r+1)], \end{array}$$

which are $=\frac{1}{2}r(r+1)(3n-4r+6)-\frac{1}{8}(r-1)(r-2)(r-3)(r-4)$,

or say $=\frac{1}{2}r(r+1)(3n-4r+6)+\frac{1}{8}(-r^4+10r^3-35r^2+50r-24)$.

I have found by an independent investigation that this value requires the correction $+\frac{1}{8}\left(r^4-8r^3+30r^2-56r+24+\dfrac{1-(-)^r 1}{2}\right)$,

and that the true value of the postulation is

$$=\frac{1}{2}r(r+1)(3n-4r+6)+\frac{1}{8}\left\{2r^3-5r^2-6r+\frac{1-(-)^r 1}{2}\right\},$$

viz., that this is the number of the conditions to be satisfied that a surface of the order n may have for an r-tuple curve three given right lines meeting in a point.

June 9th, 1870.

Professor CAYLEY, President, in the Chair.

The Hon. Sir James Cockle, F.R.S., was elected a Member.

The President read the following

Note on the Cartesian with Two Imaginary Axial Foci.

Let A, A′, B, B′ be a pair of points and anti-points;

viz., (A, A′) the two imaginary points, coordinates $(\pm\beta i, 0)$,

(B, B′) the two real points, coordinates $(0, \pm\beta)$;

and write $\rho, \rho', \sigma, \sigma'$ for the distances of a point (x, y) from the four points respectively; say

$$\rho = \sqrt{(x+\beta i)^2+y^2}, \quad \sigma = \sqrt{x^2+(y+\beta)^2},$$
$$\rho' = \sqrt{(x-\beta i)^2+y^2}, \quad \sigma' = \sqrt{x^2+(y-\beta)^2}.$$

We have $$\rho^2+\rho'^2 = 2x^2+2y^2-2\beta^2 = \sigma^2+\sigma'^2-4\beta^2,$$

$$\rho\rho' = \sqrt{(x+\beta i+yi)(x+\beta i-yi)(x-\beta i+yi)(x-\beta i-yi)} = \sigma\sigma';$$

and thence $$(\rho+\rho')^2 = (\sigma+\sigma')^2-4\beta^2,$$
$$(\rho-\rho')^2 = (\sigma-\sigma')^2-4\beta^2;$$

or say $$\rho+\rho' = \sqrt{(\sigma+\sigma')^2-4\beta^2},$$
$$i(\rho-\rho') = \sqrt{4\beta^2-(\sigma-\sigma')^2}.$$

The equation of a Cartesian having the two imaginary axial foci A, A′ is

$$(p+qi)\rho+(p-qi)\rho'+2k^2 = 0;$$

viz., this is $$p(\rho+\rho')+qi(\rho-\rho')+2k^2 = 0;$$

or, what is the same thing, it is

$$p\sqrt{(\sigma+\sigma')^2+4\beta^2}+q\sqrt{4\beta^2-(\sigma-\sigma')^2}+2k^2 = 0,$$

which is the equation expressed in terms of the distances σ, σ' from the non-axial real foci B, B′. Of course, the radicals are to be taken with the signs $\pm$. This equation gives, however, the Cartesian in combination with an equal curve situate symmetrically therewith in regard to the axis of y.

The distances σ, σ' may conveniently be expressed in terms of a single variable parameter θ; in fact, we may write

$$\pm p\sqrt{(\sigma+\sigma')^2-4\beta^2} = -k^2-k\theta,$$
$$\pm q\sqrt{4\beta^2-(\sigma-\sigma')^2} = -k^2+k\theta;$$

that is $$(\sigma+\sigma')^2-4\beta^2 = \frac{k^2}{p^2}(k+\theta)^2,$$

$$4\beta^2-(\sigma-\sigma')^2 = \frac{k^2}{q^2}(k-\theta)^2;$$

and therefore $\sigma+\sigma' = \sqrt{4\beta^2 + \frac{k^2}{p^2}(k+\theta)^2}$,

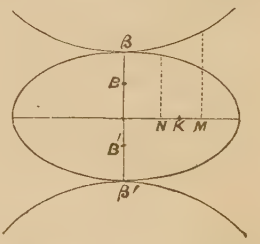
$$\sigma-\sigma' = \pm\sqrt{4\beta^2 - \frac{k^2}{q^2}(k-\theta)^2}:$$

so that, assigning to θ any given value, we have σ, σ', and thence the position of the point on the curve. We may draw the hyperbola $y^2 = 4\beta^2 + \frac{k^2}{p^2}x^2$, and the ellipse $y^2 = 4\beta^2 - \frac{k^2}{q^2}x^2$; and then measuring off in these two curves respectively the ordinates which belong to the abscissæ $k+\theta$ for the hyperbola, $k-\theta$ for the ellipse, we have the values $\sigma+\sigma'$ and $\sigma-\sigma'$, which determine the point on the curve. Considering k, p, q, β as disposable quantities, the conics may be any ellipse and hyperbola whatever, having a pair of vertices in common; and the complete construction is,—From the fixed point K in the axis of x, measure off in opposite directions the equal distances KM, KN, and take

$\sigma+\sigma'$ the ordinate at M in the hyperbola,

$\pm(\sigma-\sigma')$ „ „ N „ ellipse;

where σ, σ' denote the distances of the required point from the fixed points B and B′ respectively, the distance of each of these from the origin being $= \frac{1}{2}$ the common semi-axis.



We may imagine N travelling from one extremity of the x-axis of the ellipse to the other, the value of $\sigma+\sigma'$ will be real and greater than BB′, that of $\sigma-\sigma'$ real and less than BB′, and the point (σ, σ') will be real. The construction gives, it will be observed, the two symmetrically situated curves.

The x-semi-axis of the ellipse is $\frac{q}{k}2\beta$, and the form of the curve depends chiefly on the value of the ratio $k : \frac{q}{k}2\beta$; or what is the same thing, $k^2 : 2\beta q$. We see, for instance, that, in order that the curve may meet the axis of y in two real points between the foci, the value $\theta = -k$ must give a real value of $\sigma-\sigma'$; viz., that we must have $4\beta^2 > \frac{4k^4}{q^2}$; that is, $\beta^2q^2 > k^4$, or $k^2 < \beta q$. If k has this value, viz., $k = \frac{1}{2}\frac{q}{k}2\beta = \frac{1}{2}$ semi-axis, the curve *touches* the axis of y at the origin: if $k < \frac{1}{2}$ semi-axis, the curve cuts the axis of y in two real points between the foci; if $k > \frac{1}{2}$ semi-axis, the curve does not cut the axis of y between the foci.

Mr. Cotterill then gave an account of his paper "On the Intersection of Plane Curves." A short discussion ensued on the subject of the communication.

The following presents were made to the Society at this meeting and subsequently:—

"Journal of the Institute of Actuaries," No. lxxix., for April, 1870.

"Monatsbericht," April, May, July, 1870.

"On Reciprocal Figures, Frames, and Diagrams of Forces," by J. Clerk Maxwell, F.R.S. (Transactions of Royal Society of Edinburgh, vol. xxvi., 1870); and "On Governors," by the same (Proceedings of Royal Society, No. c., 1868): from the Author.

"Transactions of Royal Irish Academy," vol. xxiv., pts. iv., viii.—xv. inclusive.

"Annali di Matematica," Serie ii^{a}, tom. iii^{o}, fasc. 4^{o}, May, 1870.

"Annuaire de l'Académie Royale des Sciences, des Lettres, et des Beaux-Arts de Belgique, 1870," and the "Bulletins" of the same Academy, 2^{e} Serie, tom. xxvii. and tom. xxviii., 1869.

"Crelle," 71^{r} Band (4^{s} Heft) and 72^{r} Band (1^{s}, 2^{s} and 3^{s} Heft).

"Journal de l'École Impériale Polytechnique, publié par le Conseil d'Instruction," (43^{e} cahier), tom. xxvi., 1870.

"On the Contact of Conics with Surfaces," by W. Spottiswoode, F.R.S.: from the Author.

From the Smithsonian Institution, Washington, U.S.: the "Report" for 1868; the "Orbit and Phænomena of a Meteoric Fire-ball," by Dr. J. H. Coffin; the "Transatlantic Longitude as determined by the Coast Survey Expedition of 1866," by B. Apthorp Gould.

"Sugli integrali a differenziale algebrico," Memoria di L. Cremona: from the Author.

"Quelques résultats obtenus par la considération du déplacement infiniment petit d'une surface algébrique;" and "Détermination du plan osculateur et du rayon de courbure de la trajectoire d'un point quelconque d'une droite que l'on déplace en l'assujetissant à certaines conditions," by M. A. Mannheim: from the Author.

SEVENTH SESSION, 1870—1.

November 10th, 1870.

ANNUAL GENERAL MEETING, held at 22, Albemarle Street (Rooms of the Asiatic Society).

Prof. CAYLEY, F.R.S., President, in the Chair.

The Reports of the Treasurer and Secretaries were read and adopted.

From the Report of the Secretaries, it appeared that the number of members, since the last General Meeting, held November 11th, 1869, had decreased from 116 to 110.

The Obituary of the Society during the Session comprised the names of Prof. E. W. Brayley, F.R.S., and Mr. J. T. Graves, F.R.S.

The communications made to the Society during the past Session had been as follows:—

"Opposite Points on a Curve:" by Mr. T. Cotterill.

"Extension of a Theorem of Serret's:" by Mr. W. K. Clifford.

"On the Order of the Discriminant of a Ternary Form:" by Mr. S. Roberts.

"Properties of the Centro-Surface of the Ellipsoid:" by Prof. Cayley, President.

"Equations of Centres and Foci, and conditions of certain Involutions:" by Mr. J. J. Walker.

"Quartic Surfaces:" by Prof. Cayley, President.

"Note on the Evaporation of Order in Resultants:" by Mr. W. K. Clifford.

"Note on a case of Fluid Motion, and Topographical Geometry:" by Mr. J. Clerk-Maxwell.

"The Pedals of Conics:" by Mr. S. Roberts.

"Mechanical Description of a Nodal Bicircular Quartic:" by Prof. Cayley, President.

"On the Ovals of Descartes:" by Mr. S. Roberts.

"Note on the Cartesian Oval, with two imaginary axial foci:" by Prof. Cayley, President.

"On the Intersection of Plane Curves:" by Mr. T. Cotterill.

Lesser communications were made by Mr. G. O. Haulon, Mr. S.

Roberts, Prof. H. J. S. Smith, Mr. M. W. Crofton, Mr. Archibald Smith, Mr. T. Cotterill, Mr. M. Jenkins, and Dr. Henrici.

The same Mathematical Journals had been subscribed for as in the preceding Session, with the addition of the "Bulletin des Sciences Mathématiques et Astronomiques."

The Proceedings of the Society had been sent to the Royal Societies of London and of Edinburgh; the Royal Irish Academy; the Cambridge Philosophical Society; the Philosophical Society of Manchester; the Institute of Actuaries; the British Museum; l'Académie des Sciences, Paris; l'École Polytechnique; la Société Philomathique; Akademie der Wissenschaften, Berlin; il Istituto Lombardo, Milan; the Smithsonian Institution, Washington, U.S.; l'Académie Royale de Belgique; Prof. Luigi Cremona, an editor of the Annali di Matematica; Herr Borchardt, Berlin, editor of Crelle's Journal; the editor of "Nature"; the editors of the "Bulletin des Sciences Mathématiques et Astronomiques"; Drs. Ohrtmann and Müller, of Berlin, editors of the "Annals of the Progress of the Mathematical Sciences."

In return for these, the Society had received the Proceedings of the Royal Society; the Journal of the Institute of Actuaries; the Annali di Matematica; the Monatsberichte for 1870, from the Akademie der Wissenschaften; Crelle's Journal, from Herr Borchardt; "Nature," weekly, from the Editor; the Transactions of the Royal Irish Society; Annuaire et Bulletins de l'Académie Royale des Sciences, des Lettres, et des Beaux-Arts de Belgique; Journal de l'École Polytechnique; Report and Proceedings (2 Nos.) of the Smithsonian Institution, Washington.

The following presents had been made to the Society:—

Memoirs on "Cubic Surfaces," and on "The Theory of Reciprocal Surfaces," by Prof. Cayley: from the Author.

"Note on a Theorem relative to Neutral Series," by A. De Morgan, F.R.A.S.: from the Author.

"On Interpolation, Summation, and the Adjustment of Numerical Tables," and "On an Improved Theory of Annuities and Assurances," by W. S. B. Woolhouse, F.R.A.S.: from the Author.

"On the Nodal Cones of Quadrinodal Cubics, and the Zomal Conics of Tetrazomal Quartics," by the Rev. R. Townsend, F.R.S.: from the Author.

"Path of the Total Phase of the Solar Eclipse," Dec. 21—22, 1870: from the Nautical Almanac Office.

"Researches respecting Quaternions," first series, by Sir W. R. Hamilton: presented by R. Hutton, Esq.

"Sulle ventisette Rette di una Superficie del terzo Ordine:" Nota del Prof. L. Cremona: from the Author.

"On Reciprocal Figures, Frames, and Diagrams of Forces," and "On Governors," by J. Clerk-Maxwell, F.R.S.: from the Author.

"On the Contact of Conics with Surfaces," by W. Spottiswoode, F.R.S.: from the Author.

"Geometric Maps and Contributions to Kinematics," by H. Perigal, F.R.A S.: from the Author.

"Sugli integrali a differenziale algebrico," Memoria di L. Cremona: from the Author.

"Quelques résultats obtenus par la considération du déplacement infiniment petit d'une surface algébrique;" and

"Détermination du plan osculateur et du rayon de courbure de la trajectoire d'un point quelconque d'une droite que l'on déplace en l'assujetissant à certaines conditions," by M. A. Mannheim: from the Author.

A portrait of Dr. C. G. J. Jacobi: from Prof. Hirst.

The following Officers and Council were elected to serve during the following year:—

President, W. Spottiswoode, Esq., F.R.S.; Vice-Presidents, Profs. Cayley, Henrici, and H. J. S. Smith; Treasurer, Dr. Hirst, F.R.S.; Hon. Secretaries, M. Jenkins, M.A., and R. Tucker, M.A.; other Members of the Council, Messrs. W. K. Clifford, T. Cotterill, M. W. Crofton, *C. W. Merrifield, *J. F. Moulton, J. Stirling, Archibald Smith, Dr. Sylvester, and J. J. Walker. (The gentlemen marked * were not on the preceding Council list.)

Votes of thanks were carried to the Chemical Society for the use of their rooms; to Dr. Sharpey for the continued use of shelves in his room at University College; and to the retiring Members of the Council for their services.

On the motion of Dr. Hirst, seconded by Mr. Spottiswoode, it was unanimously agreed, that the Meetings of the Society should, for the ensuing Session, be held in rooms at 22, Albemarle Street, occupied by the British Association, with the sanction of the Asiatic Society. The Library of the Society to be removed to a room in the same building.

Mr. J. Hamblin Smith, M.A., of Gonville and Caius College, Cambridge, was proposed for election.

The President elect having taken the chair, Prof. Cayley read the following paper:—

Sketch of Recent Researches upon Quartic and Quintic Surfaces.

The classification of quartic surfaces is even as to its highest divisions incomplete; and it is by no means easy to make it at once exhaustive and precise; an enumeration of all the *primâ facie* possible cases would include forms which do not really exist. Thus the singular curve (if any) is of the order 1, 2, or 3—but in the case where the order is = 3, the curve, as is at once evident, cannot be a plane cubic.

nor (among other excluded forms) a system of three non-intersecting lines. And certain forms of the singular curve, *e. g.* all but one of the admissible forms of a curve of the order 3, make the surface to be a scroll, so that if (as is convenient) we wish to separate the scrolls, certain forms, otherwise admissible, must be excluded. The expression "singular" means double or cuspidal, or refers to a higher singularity, but the cases of higher singularity are very special. I will, at the cost of some inaccuracy, use the expression "nodal," as meaning, in general, double, but as including the signification "cuspidal"; and, if there are any cases of higher singularity, as extending to cases of higher singularity: and I provisionally arrange the non-scrolar quartic surfaces as follows:—

1. Without a nodal curve.
2. With a nodal line.
3. With a nodal conic, or line-pair (pair of intersecting lines).
4. With three nodal lines (not in the same plane) meeting in a point.

(Observe that the omitted cases are cases which, as I believe, ought to be omitted; thus the case of a nodal skew cubic is omitted, because the surface is then of necessity a scroll). And to these I join:—

5. The quartic scrolls;

omitting altogether the torse and cones.

The references, by the name of the author, and number (if any) of his paper, are to the subjoined list of Memoirs.

As to the scrolls, we have Cayley (3) and (4), and Cremona; the division into 12 species is, I believe, complete: see *post*, the remarks upon Schwarz's paper on quintic scrolls.

As regards the non-scrolar surfaces:—

1. Without a singular curve. The surface may be without a cnicnode (conical point), or it may have any number of cnicnodes up to 16, Cayley (7): the cases of singularity higher than a cnicnode are probably very numerous, but they have been scarcely at all examined. The memoir just referred to relates chiefly to the several cases of not more than 10 nodes; the cases of 11, 12, 13, 14, 15, 16 nodes are considered incidentally, Kummer (2), but it was not the object of his paper to make an enumeration, and there *may be* cases which are not considered; the discussion of the cases considered is very full and interesting. The case of 16 nodes is also considered, Kummer (1). As to the surface with 16 nodes, it is to be remarked that the wave-surface, or generally the surface obtained by the homographic deformation of the wave-surface—called, Cayley (1), the "tetrahedroid"—is a special form of surface with 16 nodes: its relation to the general surface is explained, Cayley (2).

2. Quartic surface with nodal line: considered incidentally, Clebsch (2) and (3). There are through the nodal line 8 planes, each meeting the surface in a line-pair: considering any 7 of these, and taking out

of each of them a line, the 7 lines are met by a conic which also meets a determinate line out of the remaining line-pair; there are thus on the surface 2^7, $= 128$, conics; viz., these form 64 pairs, each pair lying in a plane, and being the complete intersection of the surface by such plane; the number of these planes is of course $= 64$.

Although not properly included in the present case, I mention the quartic surface which is the reciprocal of the cubic surface XIX $= 12 - B_6 - C_2$, Cayley (5): the nodal curve is here an oscnodal line counting as three nodal lines.

3. Quartic surfaces with nodal conic. Such a surface may be without cnicnodes, or it may have 1, 2, 3, or 4 cnicnodes; the cases, other than that of 3 cnicodes, are mentioned, Kummer (3); but the question is examined, and the remaining case of 3 cnicnodes established, Cayley (6).

The general case of the nodal conic without cnicnodes is elaborately considered, Clebsch (1): it is shown that there are on the surface 16 lines, each meeting the conic, and which in their arrangement are strikingly analogous to the 27 lines on a cubic surface; viz., if on a cubic surface we select at pleasure any one of the 27 lines, and through this line draw a plane which besides meets the cubic surface in a conic; then, disregarding the line in question and the 10 lines which meet it, the remaining 16 lines each meet the conic, and are related to it and to each other in the same manner that the 16 lines of the quartic surface are related to the nodal conic and to each other. And the ground hereof appears, Geiser; viz., it is shown that the quartic surface with the nodal conic, is rationally transformable into a cubic surface, the 16 lines and the nodal conic corresponding respectively to the 16 lines and the conic of the cubic surface.

The several cases of 1, 2, 3, and 4 cnicnodes are considered, Korndörfer.

In the case where the nodal conic is the circle at infinity, the surfaces have been termed "anallagmatic" (perhaps "bicircular" would be a more convenient name), and a great deal has been written upon these surfaces by Moutard, Clifford, and others. Such a surface may of course have 1, 2, 3, or 4 cnicnodes; these surfaces, viz. the cnicnodal anallagmatics, in fact arise from the inversion of a quadric surface by the method of reciprocal radius vectors (that is, by the change of x, y, z into $\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$: the centre of inversion is a node on the quartic surface. If the quadric surface is a cone, there is another node, the inverse point of the vertex; if the quadric surface is one of revolution, there are two other nodes; and if it is a cone of revolution, there are three other nodes—viz., in all, four nodes. The last mentioned surface is, or includes, the Cyclide; viz., this is a quartic

surface having the circle at infinity for a nodal curve, and having besides four nodes, which are a system of skew antipoints. The surface was first considered by Dupin (Applications de Géométrie &c., 1822) as the envelope of a sphere touching three given spheres—its lines of curvature are thus circles; and the surface has been very frequently considered in reference to this property and otherwise: see Maxwell, where a classification (not quite complete) is made of the different forms of the surface, and also stereographic drawings given. It is to be observed, that one interesting form, the parabolic cyclide, is not a quartic but a cubic surface.

In the class of surfaces which have been under consideration, the cnicnodes have been points not on the nodal conic—in fact, a point on a nodal curve cannot be, properly speaking, a cnicnode, though it may be a point of higher singularity in the nature of a cnicnode; viz., there may be on the nodal curve points which, in the classification of the surfaces, must be counted as cnicnodes. Such a case presents itself in the "Conic Torus," or surface generated by the rotation of a conic about a line whether not in or in the plane of the conic. The surface has been considered, De la Gournerie (1), although more in reference to the constructions of descriptive geometry than as a theory of pure geometry, and Cayley (6). The surface has a nodal circle, and upon it two singular points, the circular points at infinity; so that it belongs to the case of a nodal conic with two cnicnodes. In the particular case where the axis of rotation is in the plane of the conic, then there are on the axis two cnicnodes; so that the case is that of a nodal conic and four cnicnodes; and when the generating conic is a circle, viz., when the surface is the ordinary torus, or anchor ring, generated by the rotation of a circle about a line in its own plane, then the nodal conic is the circle at infinity having upon it two cnicnodal points (its intersections by the planes at right angles to the axis) and the surface has also two cnicnodes on the axis: the surface, although presenting considerable peculiarity, may be regarded as a particular case of the cyclide. In reference to the plane sections of the conic torus and its various particular cases, see De la Gournerie (2); the ordinary torus has been the subject of numerous papers by Darboux and others, and possesses very interesting properties.

In connexion with the foregoing, I speak of the surfaces having a cuspidal conic: the general case is briefly referred to, Cayley (6); viz., this is the surface (AA) the equation of which is $V^2 - x^3y = 0$, and which it is shown has a reciprocal of the order 6. A special case is the surface (AB) having a reciprocal of the order 3; viz., the quartic surface is here the reciprocal of the cubic surface $XX = 12 - U_8$, Cayley (5). And it appears from the memoir last referred to, that there is

another cubic surface, $XVII = 12 - 2B_3 - C_2$, the reciprocal of which is a quartic surface having a cuspidal conic. But the theory of the quartic surfaces with a cuspidal conic has been hardly at all considered.

I do not know that anything has been done in regard to the quartic surfaces where the nodal conic becomes a line-pair; that is, where we have two intersecting nodal lines. Although not properly belonging to the case in question, I mention here the quartic surface which is the reciprocal of the cubic surface $XVIII = 12 - B_4 - 2C_2$, Cayley (5); the nodal curve consists of two intersecting lines, but one of them is tacnodal, counting as two nodal lines.

4. Quartic surface with three nodal lines (not in the same plane) meeting in a point. This is, in fact, Steiner's quartic surface; and it has been the subject of numerous investigations.

The equation of the surface may be taken to be $\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w} = 0$; and the surface thus presents itself as the reciprocal of the cubic surface $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{w} = 0$, $(XVI = 12 - 4C_2,)$ with four cnicnodes.*

It was convenient to make the foregoing enumeration before speaking of Kummer's paper (3), and of the several memoirs which relate to the Abbildung of certain quartic and quintic surfaces.

As regards Kummer's paper, the object appears by the title, viz., he considers in what cases a quartic surface has upon it a system of conics; or, what is the same thing, in what cases there is a system of planes each intersecting the surface in two conics. It is, in the first place, remarked that there is no proper quartic surface cut by every plane in a pair of conics, or even a proper quartic surface cut in a pair of conics by every plane through a fixed point. The cases considered are—I., where the planes are non-tangent planes; II., where they are single tangent planes; and III., where they are double tangent planes. The case I. is—(1) when there is a nodal conic and two cnicnodes; viz., any plane through the 2 cnicnodes gives a section with 4 nodes, therefore a pair of conics, (and the special case of 4 cnicnodes is noticed incidentally);—(2) when there is a nodal line; any plane through the nodal line besides meets the surface in a conic;—(3) when the surface has two "Selbstberührungspuncte"; viz., either of these is a point where the tangent plane is replaced by two coincident planes, and which, when a plane passes through it, gives in the section a tacnode, =2 nodes; the section by a plane through two such points, consists of two conics touching each other at the point in question: the equation of such a surface is $\phi^2 = (* \sum p, q)^4$, where ϕ is a quadric function, and p, q

* The Author exhibited, and pointed out some of the properties of, a model of Steiner's surface.

linear functions of the coordinates. In all the cases the planes pass through a fixed line, and the surface may be considered as the locus of a variable conic, the plane of which always passes through such line. II. is—(1) Steiner's surface, where every tangent plane meets the surface in a pair of conics; and (2) surface with a nodal conic and one cnicnode, where every tangent plane through the cnicnode meets the surface in a pair of conics. And III. is (1) the surface with a nodal conic, where every double tangent plane meets the surface in a pair of conics: it is shown that there are 5 quadric cones, such that a tangent plane of any one of these cones is always a double tangent plane of the surface. Or the surface is (2) a quartic scroll; any plane through two intersecting lines of the surface besides meets the surface in a conic.

It is in the paper, Cayley (6), remarked, that the quartic surface $(*\,\mathfrak{X}\,U, V, W)^2 = 0$ can also be expressed in the form $UW - V^2 = 0$; under which form the surface is seen to be the envelope of the series of quadric surfaces $(U, V, W\,\mathfrak{X}\,\theta, 1)^2 = 0$; and by reason of this property it is very easy to find the equations of the reciprocal surfaces, or plane-equations of the quartic surfaces in question. And, in the same paper, it is noticed that the surfaces of the form in question include the reciprocals of several interesting surfaces of the orders 6, 8, 9, 10, and 12; viz., order 6, parabolic ring: order 8, elliptic ring: order 9, centro-surface of paraboloid: order 10, parallel surface of paraboloid; envelope of planes through the points of an ellipsoid at right angles to the radius vectors from the centre: order 12, centro-surface of ellipsoid; parallel surface of ellipsoid.

It will be noticed that several of the papers by Clebsch and others refer in their titles to the "Abbildung" of a surface; viz., they show that a (1, 1) correspondence exists between the points of the surface and the points of a plane. The most simple instance is the quadric surface; here, taking any fixed point O on the surface, the line OP drawn to any point P on the surface meets a plane in a point P′, and the points P, P′ have, it is clear, a (1, 1)· correspondence. And, of course, to any curve on the quadric surface there corresponds a curve on the plane, and the discussion of the nature of the plane curves which correspond to the different curves on the quadric surface would constitute a theory of the Abbildung of the quadric surface.

Similarly, as remarked, Clebsch (2), for a cubic surface, taking upon it any two lines which do not meet, if from a point P on the surface we draw, meeting each of the two lines, a line to meet the plane in P′, then the point P on the cubic surface and the point P′ on the plane will have a (1, 1) correspondence; and we have thus a like theory for the cubic surface. The Abbildung of a cubic surface had been, however, previously effected by Clebsch [in the paper "Die Geometrie auf den Flächen dritter Ordnung," Crelle, t. 65 (1866), pp. 359—380], and

by Cremona, in a different and really the most simple manner,* but having a less obvious geometrical signification.

For surfaces of the higher orders, it is only certain surfaces which admit of an Abbildung, or (1, 1) correspondence of the points thereof with the points of a plane; viz. (in the same way as a plane curve, in order to its being unicursal, must have a sufficient number of nodes or cusps) a surface, in order that it may thus correspond with the plane (or say, in order that it may be unicursal), must have a sufficient singularity in the way of a nodal or cuspidal curve. The quartic and quintic surfaces considered in the enumerated memoirs are there considered for the sake of the Abbildung theory which they give rise to; whereas, in the present sketch, the Abbildung theory is considered only for the sake of the quartic and quintic surfaces to which the theory has been applied. But the methods of the theory furnish results in relation to these surfaces; and it is proper to give some account of them.

Clebsch's memoirs (2) and (3) relate to the same subject, which is elaborately treated in the latter of them: the former of them contains, however, some valuable remarks which are not reproduced in the other. In these memoirs (2) and (3), after explaining the above method of the transformation of a cubic surface by means of two of the lines thereof, the author goes on to notice that the like method is applicable to certain quartic and quintic surfaces; viz., (1) quartic surface with a nodal conic: there are here, as already mentioned, 16 lines, each meeting the conic; if, selecting any one of these, from a point P on the surface we draw, meeting the line and the conic, a line to cut the plane in P′, then the points P on the surface and P′ on the plane have a (1, 1) correspondence. (2) Quartic surface with a nodal line: as already mentioned, there are on the surface 128 conics, each meeting the nodal line; selecting any one of these, if from a point P of the surface we draw, meeting the nodal line and the conic, a line cutting the plane in P′, then the point P on the surface, and the point P′ on the plane, have a (1, 1) correspondence.

Similarly, (3), for a quintic surface having a nodal skew cubic; then if from a point P on the surface we draw, meeting the skew cubic twice, a line to cut the plane in P′, the point P on the surface and the point P′ on the plane have a (1, 1) correspondence. The nodal skew cubic may break up into a conic and line which meets it, or into

* Any method of transformation leads to an expression of the coordinates of a point on the surface as proportional to rational and integral functions of a given degree ν of the coordinates (x, y, z) of a point on the plane, and that transformation is the more simple for which ν has the smaller value: for the method of the text, the value is $\nu = 3$, but for the methods previously given by Clebsch and Cremona, it is $\nu = 2$.

three lines, two of them not meeting each other, but each met by the third line; and the like theory applies to these quintic surfaces.

It is to be noticed that (as for the cubic surface) the above methods of Abbildung, although they have the most obvious geometrical significations, are (as explained in the foregoing foot-note) not the most simple ones; but for each of the foregoing cases (1), (2), (3), the most simple transformation is established in the memoirs now under consideration. The memoir of Korndörfer, as indicated by its title, relates to the Abbildung of a quartic surface having a nodal conic and 1, 2, 3, or 4 cnicnodes.

Clebsch's paper (4) relates to the Abbildung of a quartic scroll.

As regards quintic surfaces (not being scrolls), we have, so far as I am aware, only the before-mentioned paper, Clebsch (3), relating to quintic surfaces with a nodal skew cubic; and the paper Clebsch (5), which relates to the Abbildung of a quintic surface having a nodal quadriquadric. The method employed is that of a preliminary Abbildung upon a twofold plane (2-blättrige Ebene); that is, it consists in establishing, in the first instance, a (1, 2) correspondence between the surface and the plane; and by means hereof it is shown that there exist on the surface the conics K and C presently referred to, and which give, ultimately, an ordinary Abbildung or (1, 1) correspondence of the points of the surface with those of the plane; viz., this final result is as follows:—

There is on the surface a system of conics K, such that their planes pass through a point and envelope a quadricone; and also 64 conics C each meeting each of the conics K in a single point: we select one of these and call it the conic C.

Take now the plane B_1 of a conic K_1 of the series of conics K, which plane B_1 passes, of course, through the vertex V of the cone enveloped by the planes of the conics K; viz., these planes intersect the plane B_1 in a series of lines passing through the point V.

Take a point P on the surface; this lies on a conic K meeting the conic C in a point ξ; and if we draw the line ξP to meet the plane B_1 in P′, then P on the quintic surface, and P′ on the plane B_1, will have a (1, 1) correspondence; in fact, it appears that, given P, there exists a single position of P′; and conversely, given P′, this lies on a line VP′ through which there passes the plane B_1 and one other tangent plane of the cone: this tangent plane contains a conic K meeting the conic C in a point ξ; and joining $\xi P'$, this meets the conic K in one other point P; viz., given P′, there is a single position of P; and there is thus a (1, 1) correspondence.

There are, as originally shown by Schläfli, and as further appears by my memoir on cubic surfaces, Cayley (5), 3 kinds of cubic surfaces of the class 5; viz., these are the surfaces $XIII = 12 - B_3 - 2C_2$,

$XIV = 12 - B_5 - C_2$, and $XV = 12 - U_7$; for each of these the reciprocal surface is a quintic surface of the class 3, having a nodal line and a cuspidal quartic curve. For the reciprocal of XIII, the cuspidal curve is a quadriquadric; for that of XIV, the cuspidal curve breaks up into the nodal line (viz., this is a cuspnodal line) and into a skew cubic; for that of XV, the cuspidal curve is a cuspidal quadriquadric, or curve of intersection of two quadric surfaces with singular contact.

It only remains to speak of Schwarz's memoir on quintic scrolls: it is to be remarked that the theory of scrolls is allied more closely with that of plane curves than with that of surfaces; viz., considering any plane section of the scroll, the lines of the scroll have, in general, a (1, 1) correspondence with the points of the plane section, and the scrolls of any given order are properly arranged according to the deficiency of the plane section. This is what is done by Cremona in the memoir on quartic scrolls above referred to; viz., for a quartic scroll the deficiency is either 0 or 1; and of the 12 species, there are 10 for which the deficiency is $= 0$ (or which are unicursal), and 2 for which the deficiency is $= 1$. And this is the principle of classification in Schwarz's memoir; viz., for a quintic scroll the deficiency is $= 0$, 1, or 2; the number of species established being 10, 4, and 1 for these deficiencies respectively.

List of Memoirs.

Cayley.—1. Sur la surface des Ondes. *Liouv.* t. xi., 1846, pp. 291—296.

——— 2. Sur un cas particulier de la surface du quatrième ordre avec seize points singuliers. *Crelle*, t. lxv. (1866), pp. 284—290.

——— 3. Second Memoir on Skew Surfaces, otherwise Scrolls. *Phil. Trans.*, vol. cliv. (1864), pp. 559—577.

——— 4. Third Memoir on Skew Surfaces, otherwise Scrolls. *Phil. Trans.*, vol. clix. (1869), pp. 111—126.

——— 5. Memoir on Cubic Surfaces. *Phil. Trans.*, vol. clix. (1869), pp. 231—326.

——— 6. On the Quartic Surfaces $(* \sum U, V, W)^2 = 0$. *Quarterly Math. Journal*, t. x., pp. 24—34, (1868); and t. xi., pp. 15—25, and pp. 111—113, (1870).

——— 7. A Memoir on Quartic Surfaces. *Proc. L. M. S.*, vol. iii., pp. 19—69, (1870).

Clebsch.—1. Ueber die Flächen vierter Ordnung, welche eine Doppelcurve zweiten Grades besitzen. *Crelle*, t. lxix. (1868), pp. 355—358.

——— 2. Intorno alla rappresentazione di Superficie algebriche sopra un piano. *Atti del R. Ist. Lomb.* (12 Nov. 1868), 13 p.

——— 3. Ueber die Abbildung algebraischer Flächen, insbesondere der vierten und fünften Ordnung. *Ann. Clebsch & Neumann*, vol. i. (1868), pp. 253—316.

Clebsch.—4. Ueber die ebene Abbildung der geradlinigen Flächen vierter Ordnung, welche eine Doppel curve dritten Grades besitzen. *Ann. Clebsch & Neumann* (1870), pp. 445—466.

——— 5. Ueber die Abbildung einer Classe von Flächen fünfter Ordnung. *Gött. Abh.*, t. xv., 64 p.

Cremona.—Sulle superficie gobbe di quarto grado. *Mem. di Bologna*, t. viii. (30 April, 1868), 15 p.

De la Gournerie.—1. Memoire sur la surface engendrée par la révolution d'une conique autour d'une droite située d'une manière quelconque dans l'espace. *Jour. de l'Ec. Polyt.*, t. xxiii. (1863), pp. 1—74.

——— 2. Memoire sur les lignes spiriques. *Liouv.*, t. xiv. (1869), 92 p.

Geiser.—Ueber die Flächen vierten Grades, welche eine Doppelcurve zweiten Grades haben. *Crelle*, t. lxx. (1869), pp. 249—257.

Korndörfer.—Ueber die Ebene Abbildung, &c. *Clebsch & Neumann*, t. xi.

Kummer.—1. [Surfaces of the fourth order with sixteen conical points.] *Berl. Monatsb.* (1864), pp. 246—260, and 495—499.

——— 2. Ueber die algebraischen Strahlensysteme, insbesondere über die der ersten und zweiten Ordnung. *Berl. Abh.* (1866), pp. 1—120.

——— 3. Ueber die Flächen vierten Grades auf welchen Schaaren von Kegelschnitten liegen. *Berl. Monatsb.* (July, 1863). *Crelle*, t. lxiv. (1864), pp. 66—76.

Maxwell.—On the Cyclide. *Quart. Math. Jour.*, t. ix. (1867), pp. 111—126.

Schwarz.—Ueber die geradlinigen Flächen fünften Grades. *Crelle*, t. lxvii. (1867), pp. 23—57.

December 8th, 1870.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Prof. Kanikoff was present at the Meeting.

Mr. J. Hamblin Smith, M.A., of Caius College, Cambridge, was elected a Member of the Society.

The Auditor (Mr. S. Roberts, M.A.) certified that he had examined the Treasurer's accounts, and found them perfectly correct.

Prof. Smith, V.P., made a communication to the Society on the subject of Elliptic Integrals.

Prof. Cayley, V.P., then gave an account of the following

Note on the Theory of the Rational Transformation between Two Planes, and on Special Systems of Points.

In Prof. Cremona's theory of the transformation of plane curves, the fundamental equations are taken to be

$$a_1+4a_2+9a_3+\ \ldots = n^2-1 \quad \ldots\ldots (1),$$

$$a_1+3a_2+6a_3+\ \ldots = \tfrac{1}{2}(n^2+3n)-2 \quad \ldots\ldots (2);$$

and from these we have as a consequence

$$a_2+3a_3+\ \ldots = \tfrac{1}{2}(n-1)\,(n-1) \quad \ldots\ldots (3);$$

viz., the first equation expresses that any two curves of the system intersect in a single variable point; the second, that the curves form a *reseau*, or system containing two arbitrary parameters; and the third, that the curves are unicursal.

In the equivalent theory of the rational transformation between two planes, as given in my "Memoir on the Rational Transformation between Two Spaces," we have the equation (1); but instead of the equation (2), it would *primâ facie* appear to be sufficient if we had the inequation $a_1+3a_2+6a_3+\ \ldots \leqq \ldots \tfrac{1}{2}(n^2+3n)-2$;
but on the ground there explained, the case

$$a_1+3a_2+6a_3+\ \ldots < \tfrac{1}{2}(n^2+3n)-2$$

is excluded, and we thus have the equation (2), giving with (1) the equation (3).

I believe the better course is to assume (1) and (3) as the fundamental equations, from them deducing (2); and we thus also get over a difficulty presently referred to, but which did not occur to me when the memoir was written.

In fact, starting with the equations $x' : y' : z' = X : Y : Z$ (which are to give $x : y : z = X' : Y' : Z'$), we have in the first instance the equation (1). Moreover, establishing x', y', z' a linear equation $ax'+by'+cz' = 0$, we have corresponding hereto a curve $aX+bY+cZ = 0$, and the coordinates x, y, z of a point on this curve are proportional to $X' : Y' : Z'$; that is, substituting for z' the value $-\dfrac{1}{c}(ax'+by')$, they are proportional to rational and integral (homogeneous) functions of (x', y'), that is, to rational and integral functions of the single parameter $x' : y'$; wherefore the curve $aX+bY+cZ = 0$, is unicursal; whence the equation (3). The like change may be made in the theory of the rational transformation between two spaces; and it is in this case a more important one.

The difficulty is as follows: It is not self-evident that we are at liberty to assume $a_1+3a_2+6a_3\ \ldots \leqq \tfrac{1}{2}(n^2+3n)-2$;

for imagine that we had a system of $(a_1, a_2, a_3, \ldots)$ points, such that $a_1+4a_2+\ldots$ being $=n^2-1$, and $a_1+3a_2+\ldots$ being $>\frac{1}{2}(n^2+3n)-2$, the points were such that the conditions in question (viz., the condition that the curve passes once through each of the points a_1, twice through each of the points a_2 ...) should be *less* than $a_1+3a_2+\ldots$, and in fact $=$ or $<\frac{1}{2}(n^2+3n)-2$; then the functions X, Y, Z would not of necessity be connected by a linear relation $\lambda X+\mu Y+\nu Z=0$, and the ground for the assumption in question, $a_1+3a_2+\ldots \gtreqless \frac{1}{2}(n^2+3n)-2$, would no longer exist. And except by the process now adopted of deriving the equation (2) from the equations (1) and (3), I do not know how the impossibility of such a system is to be established; viz., I do not know how we are to prove the following theorem :—There is not any system of $(a_1, a_2, a_3 \ldots)$ points, where

$$a_1+4a_2+9a_3\ldots = n^2-1,$$
$$a_1+3a_2+6a_3\ldots > \tfrac{1}{2}(n^2+3n)-2,$$

such that (a curve of the order n passing once through each point a_1, twice through each point a_2, ...) the number of conditions actually imposed on the curve is $=$ or $<\frac{1}{2}(n^2+3n)-2$.

A system of $(a_1, a_2 \ldots)$ points such that the number of actually imposed conditions is less than $a_1+3a_2+\ldots$, may be termed a special system; we have, of course, the well-known case $(a_1=n^2)$ of a system of n^2 points, such that any curve of the order n passing through $\frac{1}{2}(n^2+3n)-1$ of these passes through all the remaining points [or what is the same thing, where the number of conditions actually imposed is $=\frac{1}{2}(n^2+3n)-1$]; and we have the following special system, which presented itself to Dr. Clebsch, in his researches on the Abbildung of a quintic surface with two non-intersecting nodal lines; viz., "$a_1=12$, $a_2=2$. We may have 12 points and 2 points such that, for a quintic curve passing once through each of the 12 points and twice through each of the 2 points, the number of conditions actually imposed (instead of being $12+3\,.\,2, =18$) is $=17$." The construction is as follows: viz., Starting with the 2 points and any 10 points, we may draw a quartic passing twice through the first of the 2 points, once through the second of them, and through the 10 points; and another quartic passing twice through the second of the 2 points, once through the first of them, and through the 10 points: the two quartics intersect in the 2 points each twice, in the 10 points, and in 2 new points, forming, with the 10 points, a system of 12 points; and the first-mentioned 2 points and the 12 points form the system in question.

A more complicated case, $a_1=10$, $a_2=6$, $a_3=1$, occurs in Dr. Nöther's paper, "Ueber Flächen, welche Schaaren rationaler Curven besitzen," Leipzig, 1870. Except these two, I do not know any other

case of a special system for which $\alpha_2, \alpha_3 \ldots$ are not all $= 0$; the investigation of such systems would, I think, be very interesting.

[In my "Memoir on the Rational Transformation between two Spaces," p. 179, line 7 from the bottom, the factor $(3n-1)$ should be $(3n-2)$; and p. 180, eighth line from the bottom, the factor $3n-4r+6$ ought to have been $3n-4r+4$, viz., the whole term should be diminished by $r(r+1)$. But the "correction," sixth line from the bottom, should be increased by this same quantity, so that the value of the "postulation," as given fourth line from the bottom, is correct.]

A short discussion took place on the above paper; and then Professor Cayley gave an account of the following paper:—

A Second Memoir on Quartic Surfaces.

In my Memoir on Quartic Surfaces, *ante* pp. 19—69, although remarking (See No. 79) that the identification was not completely made out, I tacitly assumed that the symmetroid and the decadianome (each of them a quartic surface with 10 nodes) were in fact identical. There is yet a good deal which I cannot completely explain; but the truth appears to be, that the decadianome includes two cases of coordinate generality, say the sextic decadianome, and the bicubic decadianome = symmetroid: viz., in the first of these the circumscribed cone, having for vertex any one of the 10 nodes, is a proper sextic cone with 9 double lines; in the second it is a system of two cubic cones, intersecting, of course, in 9 lines, which are double lines of the aggregate sextic cone: or, in the notation of the Table No. 11, in the case of the sextic decadianome, the circumscribed cones are each of them 6_9; in that of the bicubic dianome = symmetroid, they are each of them (3, 3). We thus arrive at a very remarkable system of 10 points in space, viz., giving the name "ennead" to any 9 points *in plano*, which are the intersections of two cubic curves, or to any 9 lines through a point which are the intersections of two cubic cones; the 10 points in space are such that, taking any one whatever of them as vertex, and joining it with the remaining points, the 9 lines form an ennead. I purpose in the present short Memoir to consider the theories in question; the paragraphs are numbered consecutively with those of the Memoir on Quartic Surfaces.

Plane Sextic Curve with 9 Nodes.

110. A sextic curve contains 27 constants; and the number of conditions to be satisfied in order that a given point may be a node is = 3. Hence it would at first sight appear that the curve could be found so as to have 9 given nodes; this would be 9×3, $= 27$ condi-

tions, or the curve would be completely determinate. But observe that through the 9 given points we have a determinate cubic curve $U=0$; we have therefore $U^2=0$ a sextic curve, and the only sextic curve with the 9 given nodes; that is, there is not in a proper sense any sextic curve with the 9 given nodes. The number of given nodes is thus $=8$ at most.

111. The sextic curve with 8 given nodes should contain $27-3.8=3$ constants. We may through the 8 given points draw the two cubics $P=0$, $Q=0$; and we have then $(a, b, c\chi P, Q)^2=0$, a bicubic, or improper sextic curve having the 8 nodes, and also a ninth node, viz., the remaining point of intersection of the two cubic curves, or say the remaining point of the ennead. Hence if $\nabla=0$ be any particular sextic curve having the 8 given nodes, we have

$$(a, b, c\chi P, Q)^2+\theta\nabla=0$$

a proper sextic curve having the 8 given nodes; and this, as containing the right number $(=3)$ of constants, will be the general sextic curve having the 8 given nodes.

112. There will be a ninth node if $\theta=0$; viz., the curve is then $(a, b, c\chi P, Q)^2=0$, a bicubic, or improper sextic curve, having for nodes the 9 points of the ennead. Observe that the ninth node is here a point completely and uniquely determined by means of the given 8 nodes. Moreover the number of constants is $=2$, so that we have here a general (improper) solution of the question of finding a sextic curve with 9 nodes, 8 of them given.

113. But if θ is not $=0$, then the ninth node must be a point on the curve $J(P, Q, \nabla)=0$; viz., this is a curve of the order 9, determined by means of the given 8 points; say it is the "dianodal curve" of these 8 points, and, as is easy to see, it has each of these 8 points for a node. The ninth node of the sextic may be any point whatever on the dianodal curve; and regarding it as a given point, the sextic will still contain 1 constant; that is, we have the general solution of the problem of finding a sextic curve with 9 nodes, 8 of them given, and the 9th a given point on the dianodal curve.

114. So long as the 8 points are arbitrary, the dianodal curve does not pass through the 9th point of the ennead, and the two cases above considered are mutually exclusive. It will be noticed how closely analogous this theory of the plane sextic with 9 nodes, is to that of the quartic surface with 8 nodes.

115. Of course, instead of the plane sextic curve, we may have a sextic cone; such a cone has at most 8 given double lines; and if there be a 9th double line, then there are the two cases of coordinate generality; viz., (1), the new double line is the ninth line of the ennead, the cone being in this case not a proper sextic cone, but a bicubic cone; (2), the new double line may be any line whatever on the dianodal cone, (cone of the order 9 determined by the 8 given

lines, and having each of these for a double line,) and regarding it as a given line on the dianodal cone, the sextic cone contains 1 constant.

Each circumscribed cone of the Symmetroid is (3, 3).

116. Using (x, y, z, w) as current coordinates of a point of the symmetroid, I take S, T, U, V to be quadric functions of the coordinates $(\alpha, \beta, \gamma, \delta)$; the equation of the symmetroid is therefore given by

$$x\,\mathrm{S}+y\,\mathrm{T}+z\,\mathrm{U}+w\,\mathrm{V}=\text{cone},$$

and the nodes thereof are determined by

$$x\,\mathrm{S}+y\,\mathrm{T}+z\,\mathrm{U}+w\,\mathrm{V}=\text{plane-pair}.$$

Suppose that a node is $(x=0, y=0, w=0)$; the condition for this is $\mathrm{V}=$plane-pair; and we may without loss of generality write $\mathrm{V}=\gamma^2+\delta^2$. Hence, putting for shortness $\Theta=x\mathrm{S}+y\mathrm{T}+z\mathrm{U}$, that is Θ a quadric function $(a, b, c, d, f, g, h, l, m, n)(\alpha, \beta, \gamma, \delta)^2$, wherein the co-efficients $a, b, \ldots$ are arbitrary linear functions of (x, y, z), but not containing w, the equation of the symmetroid is given by

$$\Theta+w\,(\gamma^2+\delta^2)=\text{cone}.$$

117. It follows that the equation is

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c+w, & n \\ l, & m, & n, & d+w \end{vmatrix} = 0;$$

viz., this is $\quad \nabla+w\,(\delta_c+\delta_d)\,\nabla+\tfrac{1}{2}w^2\,(\delta^c+\delta_d)^2=0,$

where ∇ denotes the foregoing determinant, writing therein $w=0$. Or, observing that ∇ contains c and d each only linearly, the equation may be written $\quad \nabla+w\,(\delta_c+\delta_d)\,\nabla+w^2\,\delta_c\delta_d\nabla=0,$

which is a quartic surface having, as it should have, the point (0, 0, 0) for one of its ten nodes.

118. The equation of the circumscribed cone is

$$(\delta_c\nabla+\delta_d\nabla)^2-4\nabla\,.\,\delta_c\delta_d\nabla=0;$$

or, what is the same thing, it is

$$(\delta_c\nabla-\delta_d\nabla)^2+4\,(\delta_c\nabla\,.\,\delta_d\nabla-\nabla\,.\,\delta_c\delta_d\nabla)=0.$$

But we have identically

$$\delta_c\nabla\,.\,\delta_d\nabla-(\tfrac{1}{2}\delta_n\nabla)^2=\nabla\,.\,\delta_c\delta_d\nabla;$$

so that the equation is

$$(\delta_c\nabla-\delta_d\nabla)^2+(\delta_n\nabla)^2=0;$$

a sextic cone breaking up into the two cubic cones

$$\delta_c\nabla-\delta_d\nabla\pm i\delta_n\nabla=0,$$

so that the cone is (3, 3). And since clearly the point (0, 0, 0) may be regarded as representing any one whatever of the 10 nodes, it follows that for any node whatever of the symmetroid, the circumscribed cone is (3, 3), so that, as stated above, bicubic decadianome = symmetroid.

Deductions from the foregoing theory.

119. Referring to No. 85 of the original memoir, it appears that, with 6 given points as nodes, we can actually find for the symmetroid an equation containing 6 constants. I cannot discover any ground for doubting that 3 of these may be determined so as to give to the symmetroid a seventh given node; and I therefore assume that with 7 given points as nodes, an equation can be found with 3 constants. The symmetroid is certainly not octadic, hence the eighth node must lie on the dianodal surface of the 7 given points. I can discover no ground for doubting but that two of the constants may be determined so that the eighth node shall be any given point whatever on the dianodal surface of the 7 points; and (this being so) that further the remaining constant may be determined so that the ninth node shall be any given point whatever on the dianodal curve of the 8 points. But if all this be so, the consequence is very remarkable; the tenth node is not any one whatever of the 22 dianodal centres of the 9 points, but it is a uniquely determinate "enneadic centre," viz., we must have the following theorem:—

120. "Take any 7 points; an eighth point at pleasure on the dianodal surface of the 7 points; a ninth point at pleasure on the dianodal curve of the 8 points. In the system of 9 points so determined, take any one as vertex, and joining it with the remaining 8, construct the ninth line of the ennead. Performing this construction with each of the 9 points successively as vertex, we obtain 9 lines passing through the 9 points respectively. These 9 lines meet in a point which is the 'enneadic centre' of the 9 points: and further, the 10 points form a completely symmetrical system, so that each one of them is the enneadic centre of the remaining 9."

121. Assuming that the 9 lines do intersect so as to give rise to an enneadic centre, there is no difficulty in conceiving that the loci, which by their intersection determine the dianodal centres, do each of them pass through the enneadic centre; so that this enneadic centre counts once or more among the dianodal centres, and the number of proper dianodal centres, instead of being $=22$, will be suppose $=22-\omega$, and if, further, the 9 points, together with the enneadic centre, are the nodes of a symmetroid, but the 9 points together with any one of the $22-\omega$ dianodal centres are the nodes of a sextic decadianome, then we must also have as follows:

122. "Considering any 9 points as above; taking any one as vertex, and joining it with the remaining 8, these 8 lines determine a dianodal ninthic cone. We have thus 9 dianodal cones, which cones pass all of them through the same $22-\omega$ points."

123. I am not able to verify these theorems *à posteriori*. It appears

to me that the theorem in regard to the enneadic centre subsists for a system of 9 points such as referred to in the statement; but that if by possibility the statement be too general, the theorem must, at all events, subsist for a more special system of 9 points; and that there certainly exist systems of 10 points, such that each 9 of the points have as an enneadic centre the tenth point. [I have since ascertained that if a quartic surface with 10 nodes has a single node (3, 3), the surface is a symmetroid; whence, by what precedes, the remaining nine nodes are each of them (3, 3). Added 25th March.]

124. I notice, as a subject of investigation, the following system of correspondence; viz., given any 8 points in space: then to every point in space corresponds a line through this point: viz., the ninth line of the ennead obtained by joining the point with the 8 given points respectively, and to each line in space a point or points on the line: viz., the point or points for each of which the line is the ninth line of the ennead obtained by joining the point with the eight given points respectively.

Dr. Hirst next entered into an explanation of his paper "On the Polar Correlation of two Planes, and its connection with their Quadric Correspondence."

Profs. Cayley, Smith, Mr. Cotterill, and Dr. Hirst took part in the discussions on the papers.

The following presents were made to the Library:—

"On the determination of the Orbit of a Planet from three Observations." By A. Cayley, F.R.S.: from the author.

"Géométrie Supérieure." By M. Chasles: from the President.

Jan. 12*th,* 1871.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Mr. R. B. Hayward, M.A., late Fellow of St. John's College, and Mathematical Master at Harrow, and the Rev. J. Wolstenholme, M.A., Fellow of Christ's College, were proposed for election.

Mr. Walker gave an account of the following paper:—

On Systems of Tangents to Plane Cubic and Quartic Curves.

1. By a "System of Tangents" I understand all the tangents which can be drawn from a point $(x_1 y_1 z_1)$ to a proper plane curve (U). According to Dr. Salmon's notation, the equation to the system of tangents drawn to a cubic curve is obtained by equating with zero the discriminant of the binary cubic in λ, μ, (Higher Plane Curves, Ed. 1, p. 68.)

$$U\lambda^3 + \Delta_1\lambda^2\mu + \Delta\lambda\mu^2 + U_1\mu^3,$$

where Δ_1 stands for $x_1\frac{dU}{dx}+y_1\frac{dU}{dy}+z_1\frac{dU}{dz}$, and Δ for

$$x_1^2\frac{d^2U}{dx^2}+y_1^2\frac{d^2U}{dy^2}+z_1^2\frac{d^2U}{dz^2}+2y_1z_1\frac{d^2U}{dy\,dz}+2z_1x_1\frac{d^2U}{dz\,dx}+2x_1y_1\frac{d^2U}{dx\,dy};$$

while U_1 is the result of substituting x_1, y_1, z_1 for x, y, z respectively in U. It will be convenient, in what follows, to use some symbol to represent the *operator* $x_1\frac{d}{dx}+\ldots$ apart from its subject. Supposing then

$$D_1=x_1\frac{d}{dx}+y_1\frac{d}{dy}+z_1\frac{d}{dz},$$

the cubic above in λ, μ would be

$$U\lambda^3+D_1U\lambda^2\mu+\tfrac{1}{2}D_1^2U\lambda\mu^2+U_1\mu^3.$$

2. As all the curves, which I shall have occasion to notice in this short paper, will be expressed in terms of U, D_1U, ... U_1, I have, at the kind suggestion of Prof. Cayley, substituted for these cumbrous symbols the single letters a, b, c, d with numerical coefficients; viz., in the case of cubic curves,

$$U=a,$$
$$D_1U=3b,$$
$$\tfrac{1}{2}D_1^2U=3c,$$
$$U_1=d;$$

so that a is the given cubic function of xyz, b a quadric, c a linear function, and d a constant. The cubic in λ, μ will now take the familiar form $a\lambda^3+3b\lambda^2\mu+3c\lambda\mu^2+d\mu^3$,
the discriminant of which is equal to

$$(ad-bc)^2-4(ac-b^2)(bd-c^2) \quad \ldots\ldots (1);$$

or, after multiplication by d^2,

$$(ad^2-3bcd+2c^3)^2+4(bd-c^2)^3 \quad \ldots\ldots (2).$$

The geometrical interpretation of this discriminant has, as far as I am aware, only been noticed hitherto when it is arranged in the form

$$(ad^2-6bcd+4c^3)a+(4bd-3c^2)b^2 \quad \ldots\ldots (3).$$

3. From (1) it follows that the equation to the six tangents from (x, y, z) to the cubic U or a, may be thrown into the form

$$u_3^2-4u_4u_2=0 \quad \ldots\ldots (4),$$

the suffixes indicating the order of the functions in $x\,y\,z$; and where

$$u_4=ac-b^2 \quad \ldots\ldots (5),$$
$$u_3=ad-bc \quad \ldots\ldots (6),$$
$$u_2=bd-c^2 \quad \ldots\ldots (7).$$

Operating with $x_1\frac{d}{dx}+y_1\frac{d}{dy}+z_1\frac{d}{dz}$, or D_1,

$$D_1a=D_1U=3b,$$
$$D_1b=\tfrac{1}{3}D_1^2U=2c,$$
$$D_1c=\tfrac{1}{6}D_1^3U=d=U_1;$$

consequently $D_1u_4 = aD_1c + cD_1a - 2bD_1b$
$= ad + 3bc - 4bc = ad - bc = u_3.$

Again $D_1u_3 = dD_1a - bD_1c - cD_1b$
$= 3bd - bd - 2c^2 = 2(bd - c^2) = 2u_2.$

Lastly $D_1u_2 = dD_1b - 2cD_1c = 2cd - 2cd = 0.$

If $a, b, c, (D_1U), \ldots$ represent the results of substituting x_1, y_1, z_1 for x, y, z respectively in $a \ldots D_1U \ldots$, then, as is well known,

$$a_1 = U_1 = d,$$
$$b_1 = \tfrac{1}{3}(D_1U)_1 = U_1 = d,$$
$$c_1 = \tfrac{1}{6}(D_1^2U)_1 = U_1 = d,$$

from which it at once appears that each of the three curves $u_4 = 0$, $u_3 = 0$, $u_2 = 0$ passes through the point $x_1\,y_1\,z_1$ from which the tangents are drawn.

Since $(x_1\,y_1\,z_1)$ is a point on u_2, and D_1u_2 vanishes identically, u_2 represents a pair of right lines drawn through that point; and since $D_1u_4 = u_3$, $D_1u_3 = (2)u_2$, $(x_1\,y_1\,z_1)$ is a double point on u_4 and its first polar u_3, while u_2 is the pair of tangents to each of those curves at the double point.

4. The curves u_4, u_3 having a common double point and common tangents at that point, will meet again in six points, and the form of (4) shows that *at these six points the six tangents to* U *drawn from* $(x_1\,y_1\,z_1)$ *touch the quartic* u_4.

Further, the form of u_4 (5) shows that it touches U in the six points in which the latter is met by its first polar b; and, therefore, that u_4, U have the same six points of contact with their common tangents. Otherwise it appears from (6) that the nine points of intersection of U and u_3 lie on the conic b, and the line c.

The conic b meets u_4 in two other points besides those referred to above. Equation (5) shows that the line c is a double tangent to u_4 at those points; while (7) shows that at the same two points u_4 is met again by the two tangents at its double point $(x_1\,y_1\,z_1)$, which are also tangents to the conic b.

When the point $(x_1\,y_1\,z_1)$ is on U, this becomes a point in which two branches of u_4 touch, c being the tangent at that point both to U and u_4, which touch in four other points lying on the conic b.

(5.) The form (2) of the discriminant gives the equation to the six tangents from $(x_1\,y_1\,z_1)$ to U in the shape

$$v_3^2 + 4u_2^3 = 0,$$

where $v_3 = ad^2 - 3bcd + 2c^3,$

and, as before, $u_2 = bd - c^2;$

provided the point $x_1\,y_1\,z_1$ be not taken on the curve U.

Of course $(x_1\,y_1\,z_1)$ is a point on v_3. Moreover,

$$D_1v_3 = d^2D_1a - 3bdD_1c - 3cdD_1b + bc^2D_1c;$$

but $D_1 a = 3b$, $D_1 b = 2c$, $D_1 c = d$, whence

$$D_1 v_3 \equiv 0,$$

so that v_3 represents three right lines through the point $(x_1 y_1 z_1)$ meeting U on the conic $3bd - 2c^2$ and the line c. Being drawn from a double point on u_3, u_4, they meet the former curve again in three points only,—viz., those common to u_3, U, and c,—a fact verified by the identity

$$du_3 \equiv v_3 - 2cu_2;$$

while they meet u_4 in six points, other than $x_1 y_1 z_1$.

Since $$cv_3 \equiv d^2(ac - b^2) + (bd - c^2)(bd - 2c^2),$$

i.e., $$\equiv d^2 u_4 + u_2 (bd - 2c^2),$$

the six points just mentioned lie on the conic $bd - 2c^2$, which has c as its chord of double contact with b.

6. In the next place let U represent a proper plane quartic curve; then the equation to the twelve tangents from $(x_1 y_1 z_1)$ will be obtained by equating with zero the discriminant of

$$(a, b, c, d, e\,\lambda, \mu)^4,$$

where

$$a = \mathrm{U},$$
$$4b = D_1 \mathrm{U},$$
$$6c = \tfrac{1}{2} D_1^2 \mathrm{U},$$
$$4d = \tfrac{1}{6} D_1^3 \mathrm{U},$$
$$e = \mathrm{U}_1;$$

D_1, as before, standing for the operator $x_1 \frac{d}{dx} + y_1 \frac{d}{dy} + z_1 \frac{d}{dz}$, and U_1 for the result of substituting x_1 for x, y_1 for y, and z_1 for z, in U. Of course b will be a cubic, c a quadratic, and d a linear function of xyz, when a or U is a quartic.

If the standard form of the discriminant of $(a, b, c, d, e\,\lambda, \mu)^4$, viz. $\mathrm{I}^3 - 27\mathrm{J}^2$,

where

$$\mathrm{I} = ae - 4bd + 3c^2,$$
$$\mathrm{J} = ace + 2bcd - ad^2 - b^2 e - c^3,$$

be transformed into $\mathrm{IK} - 3\mathrm{J}'^2$, by writing

$$\mathrm{J}' = 3\mathrm{J} + c\mathrm{I} = 4ace + 2bcd - 3ad^2 - 3b^2 e,$$
$$\mathrm{K} = \mathrm{I}^2 + 3c(6\mathrm{J} + c\mathrm{I}),$$

then the equation to the twelve tangents from $(x_1 y_1 z_1)$ to U may be exhibited under either of the forms

$$u_4 u_8 - 3u_6^2 = 0 \quad \ldots\ldots (8),$$

or

$$u_4^3 - 27 v_8^2 = 0 \quad \ldots\ldots (9),$$

where

$$u_4 = ae - 4bd + 3c^2 \quad \ldots\ldots (10),$$
$$v_6 = ace + 2bcd - ad^2 - b^2 e - c^3 \quad \ldots\ldots (11),$$
$$u_6 = 3v_6 + cu_4 \quad \ldots\ldots (12),$$
$$u_8 = u_4^2 + 3c(6v_6 + cu_4) \quad \ldots\ldots (13),$$

the suffixes indicating the order of the functions in x, y, z.

7. If a_1, b_1, c_1, d_1 indicate the results of the substitution of x_1 for x, y_1 for y, and z_1 for z, in $a \ldots d$ respectively; then, since

$$a = \mathrm{U}, \qquad a_1 = \mathrm{U}_1 = e;$$

since $\quad 4b = \mathrm{D}_1\mathrm{U}, \qquad b_1 = \mathrm{U}_1 = e;$

since $\quad 6c = \tfrac{1}{2}\mathrm{D}_1^2\mathrm{U}, \qquad c_1 = \mathrm{U}_1 = e;$

and since $\quad 4d_1 = \tfrac{1}{6}\mathrm{D}_1^3\mathrm{U}, \qquad d_1 = \mathrm{U}_1 = e.$

It is easily verified, therefore, that u_4, v_6 each pass through $(x_1 y_1 z_1)$.

Again,

$$\mathrm{D}_1 a = \mathrm{D}_1\mathrm{U} = 4b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$

$$\mathrm{D}_1 b = \tfrac{1}{4}\mathrm{D}_1^2\mathrm{U} = 3c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15),$$

$$\mathrm{D}_1 c = \tfrac{1}{12}\mathrm{D}_1^3\mathrm{U} = 2d \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16),$$

$$\mathrm{D}_1 d = \tfrac{1}{24}\mathrm{D}_1^4\mathrm{U} = \mathrm{U}_1 = e \quad \ldots\ldots\ldots\ldots\ldots\ldots (17).$$

From these relations it is readily verified that

$$\mathrm{D}_1 u_4 \equiv 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18),$$

$$\mathrm{D}_1 v_6 \equiv 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (19),$$

consequently u_4, v_6 represent systems of four and six right lines respectively drawn through $(x_1 y_1 z_1)$.

From (12), (19), (16), it follows at once that

$$\mathrm{D}_1 u_6 = 2d u_4;$$

and from (17), (18), $\quad \mathrm{D}_1^2 u_6 = 2e u_4,$

$$\mathrm{D}_1^3 u_6 \equiv 0;$$

consequently $(x_1 y_1 z_1)$ is a quadruple point on u_6, and u_4 is the system of four tangents to u_6 at this point. Further the form of u_6, viz. $3v_6 + cu_4$, shows that the four lines u_4 meet u_6 only on points common to u_4 and v_6, *i.e.*, only at $(x_1 y_1 z_1)$, which is therefore a point of inflexion on each of the four branches of u_6 meeting at that point.

Also, (13), (16), (18), (19),

$$\begin{aligned}\mathrm{D}_1 u_8 &= 6d\,(6v_6 + cu_4) + bcdu_4 \\ &= 12d\,(3v_6 + cu_4) = 12d u_6 \quad \ldots\ldots\ldots\ldots\ldots (20);\end{aligned}$$

consequently u_8 has its first polar with respect to $(x_1 y_1 z_1)$, through which it passes, the sextic u_6 and the line d. The point $(x_1 y_1 z_1)$ is therefore a quadruple inflexion point on u_8 also, and the four lines u_4 are the tangents to u_8 at that point, each touching one of its four branches in three consecutive points. The form of u_8, viz.

$$u_8 = u_4^2 + 3c\,(6v_6 + cu_4),$$

shows that the eight points, other than $(x_1 y_1 z_1)$, in which its four tangents u_4 meet it, lie on the conic c, which has octuple contact with u_8 at these points.

8. The curves u_6, u_8 having $(x_1 y_1 z_1)$ as a common quadruple-inflexion point, and common tangents at that point, will meet again in only twenty-four points, and the equation (8),

$$u_4 u_8 - 3u_6^2 = 0,$$

one of the forms of the equation of the twelve tangents drawn from $(x_1\, y_1\, z_1)$ to U, shows that *at those points the same twelve lines are double tangents to* u_8.

It may be further shown that these twelve double tangents to u_8 touch this curve each once at its point of contact with U; or, in other words, that u_8 touches U at the twelve points of contact with the latter curve of their common tangents, the other eight points of intersection of u_8 and U lying on a conic.

For all values of the variables which satisfy $U = 0$, *i.e.* $a = 0$, see (10), (11),

$$u_4 = 3c^2 - 4bd,$$
$$v_6 = 2bcd - b^2e - c^3;$$

whence, and from (12), $u_6 = (2cd - be)\, b$;

consequently the intersections of U, or a, and its first polar b lie on u_6. Again, generally, (13), $u_8 = u_4 (u_4 + 3c^2) + 18cv_6$;

so that for all values of x, y, z, which satisfy U or $a=0$, substituting the values of u_4, v_6 just above,

$$u_8 = 2\{(3c^2 - 4bd)(3c^2 - 2bd) + 18bc^2d - 9b^2ce - 9c^4\}$$
$$= 2(8d^2 - 9ce)\, b^2;$$

consequently, at the twelve points in which U meets its first polar b, it touches u_8; the remaining eight points of intersection of U with u_8 lying on the conic $8d^2 - 9ce$.

9. The four tangents u_4 to u_8 at the quadruple-inflexion point $(x_1\, y_1\, z_1)$ count as twenty-four of the fifty-six tangents which can, in general, be drawn from a point to u_8. Of the other thirty-two, twenty-four, viz. the twelve double tangents, have been accounted for. In order to find the remaining eight, I have thought it worth while to investigate independently the equation to the thirty-six tangents which can be drawn from $(x_1\, y_1\, z_1)$ considered as a quadruple point only, as may be done without difficulty. It is necessary to form the discriminant of the quartic

$$u_8\lambda^4 + \mathrm{D}_1 u_8 \lambda^3\mu + \tfrac{1}{2}\mathrm{D}_1^2 u_8 \lambda^2\mu^2 + \tfrac{1}{6}\mathrm{D}_1^3 u_8 \lambda\mu^3 + \tfrac{1}{24}\mathrm{D}_1^4 u_8 \mu^4 \ldots\ldots (21).$$

Now $u_8 = u_4^2 + 3c(6v_6 + cu_4)$,

and, (20), $\mathrm{D}_1 u_8 = 12d(3v_6 + cu_4)$;

whence, (16)—(19), $\tfrac{1}{2}\mathrm{D}_1^2 u_8 = 6\{e(3v_6 + cu_4) + 2d^2u_4\}$,

$$\tfrac{1}{4}\mathrm{D}_1^3 u_8 = 12deu_4,$$
$$\tfrac{1}{24}\mathrm{D}_1^4 u_8 = 3e^2u_4.$$

Comparing the quartic (21) in λ, μ with $(\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}\,\mathrm{X}\,\lambda, \mu)^4$,

$$\mathrm{A} = u^2 + 3c(6v + cu),$$
$$\mathrm{B} = 3d(3v + cu),$$
$$\mathrm{C} = e(3v + cu) + 2d^2u,$$
$$\mathrm{D} = 3deu,$$
$$\mathrm{E} = 3e^2u,$$

the suffixes to v_6 and u_4 being dropped for shortness. Hence

$$AE-4BD+3C^2 = 3\{e^2(u^3-27v^2)+4(ceu-d^2u+3ev)^2\},$$

and $$ACE+2BCD-AD^2-B^2E-C^3 =$$
$$\{3e^2(u^3-27v^2)+8(ceu-d^2u+3ev)^2\}(ceu-d^2u+3ev).$$

10. If for shortness we write

$$l = e^2(u_4^3-27v_6^2),$$
$$m = (ce-d^2)u_4+3ev_6,$$

the equation to the thirty-six tangents from $(x_1\,y_1\,z_1)$ to u_8 will therefore be $$(l+4m^2)^3-(3l+8m^2)^2m^2 = 0,$$

which reduces identically to $l^2(l+3m^2) = 0$.

Restoring their values for l, m, this equation becomes identically

$$u_4(u_4^3-27v_6^2)^2[e^2\{u_4^2+3c(6v_6+cu_4)\}-3d^2\{2e(3v_6+cu_4)-d^2u_4\}] = 0;$$

i.e., (13), (12), $$u_4(u_4^3-27v_6^2)^2\{e^2u_8-3d^2(2eu_6-d^2u_4)\} = 0 \quad \ldots\ldots (22).$$

The first factor enters in consequence of the four lines u_4 meeting the curve each in two points consecutive, on each branch, to the quadruple point. The second factor is the representative of the twelve double tangents; viz., the tangents to U from $(x_1\,y_1\,z_1)$: lastly, the third factor represents the remaining eight tangents, touching u_8 at the eight points in which it is met by the line d, which appeared before as a factor in its first polar (20). These last eight tangents being drawn from a quadruple point will intersect u_8 again each in two points only lying on the sextic $$2eu_6-d^2u_4 = w_6 = 0.$$

This sextic w_6 has—as appears from its form—sextuple contact with u_6 at the six points in which they are both met by the line d. In fact w_6 has $(x_1\,y_1\,z_1)$ also as a quadruple-inflexion point, u_4 being also the four tangents to it at that point. Consequently w_6 meets u_8 in twenty-four points other than $(x_1\,y_1\,z_1)$; viz., in the eight common to u_8, u_4 and the conic c, besides the sixteen above alluded to. This appears from the identity $$eu_8-3cw_6 \equiv u_4\{eu_4-3c(ce-d^2)\}.$$

11. It appears from (13) that the six lines v_6 meet u_8 in twenty-four points, other than $(x_1\,y_1\,z_1)$, lying on the quartic

$$u_4+3c^2 = 0,$$

the other eight points common to this quartic, and u_8 being the eight common to the four lines u_4 and the conic c, which in fact are the eight points of contact with the quartic of its four double tangents u_4.

12. When the point $(x_1\,y_1\,z_1)$ falls on the curve U, U_1 or $e = 0$, and

$$u_4 = -4bd+3c^2,$$
$$v_6 = 2bcd-ad^2-c^3,$$
$$u_6 = (2bc-3ad)d,$$
$$u_8 = 2(8b^2-9ac)d^2.$$

In this case, then, u_8 breaks up into the square of the line d—the tangent to U at the point $(x_1 y_1 z_1)$—and the sextic

$$v_6 = 8b^2 - 9ac,$$

while, correspondingly, u_6 breaks up into the line d and the quintic

$$v_5 = 2bc - 3ad,$$

which may be verified to be the first polar of v_6. For since

$$\mathrm{D}_1 a = 4b, \quad \mathrm{D}_1 b = 3c, \quad \mathrm{D}_1 c = 2d, \quad \mathrm{D}_1 d = e,$$

$$\begin{aligned} \mathrm{D}_1 v_6 &= 16b\mathrm{D}_1 b - 9a\mathrm{D}_1 c - 9c\mathrm{D}_1 a \\ &= 48bc - 18ab - 36bc = 6\,(2bc - 3ad) = 6v_5. \end{aligned}$$

Again,

$$\begin{aligned} \mathrm{D}_1 v_5 &= 2b\mathrm{D}_1 c + 2c\mathrm{D}_1 b - 3a\mathrm{D}_1 d - 3d\mathrm{D}_1 a \\ &= 4bd + bc^2 - 12bd = 2\,(3c^2 - 4bd) = 2u_4; \end{aligned}$$

so that v_6, v_5 have $(x_1 y_1 z_1)$ as a common quadruple point and four common tangents at that point; consequently they meet again in ten points only. The ten tangents which can be drawn from $(x_1 y_1 z_1)$ to U or a touch v_6 at the same points, as is evident from the form of v_6; and being drawn from a quadruple point do not meet that curve again.

Prof. Cayley made a few remarks on Mr. Walker's paper, in the course of which he suggested the alteration subsequently adopted by the Author.

Mr. S. Roberts then read a paper

On the Order and Singularities of the Parallel of an Algebraical Curve.

1. A Parallel of a curve is variously defined as the envelope of a circle of constant radius which touches the given curve, or has its centre thereon; and, as the locus of the centre of a circle of constant radius which touches the given curve.

The most obvious property of a Parallel is, that among the normal distances of any point on it from the primitive curve, there is always one of given length, which I call the Modulus of the Parallel. Moreover, if at any point on the primitive we draw a normal, two points will be determined on the parallel by taking upon the normal lengths equal to the Modulus in opposite directions from the point; and the normals to the Parallel at these points coincide with the corresponding normal of the Primitive.

The parallel, therefore, has the same normals as the primitive, but each is normal at two points. The latter curve is a parallel to itself to the modulus 0, but we shall have reason to remark that as the normal distances which determine points on the parallel are measured in two directions, on opposite sides of the corresponding branch of the primitive, this, as parallel to itself, must be reckoned twice.

Since the primitive is included in the family of parallels, they may

be, in some points of view, preferably described as the system of involutes having a given evolute.

The variation of the modulus may affect the characteristics as well as the general shape and dimensions of the parallel. Hence the results which we shall be able to obtain must, in general, be held to refer to a parallel whose modulus is general.

2. Let $\phi(x, y) = 0$ be the equation of the primitive. The envelope of the circles $(x-\alpha)^2+(y-\beta)^2 = r^2, \quad \phi(\alpha, \beta) = 0$ .....(a) is a parallel to the modulus r. Disregarding mere position, we may say that the envelope of a circle translated without rotation, so that a point of it describes the curve $\phi(x, y) = 0$ will be parallel to the curve with the radius of the circle as modulus.

If we write α' for $x-\alpha$, β' for $y-\beta$, we get the system of curves,

$$\phi(x-\alpha', y-\beta') = 0,$$
$$\alpha'^2+\beta'^2-r^2 = 0,$$

and the envelope of these is the same parallel as before. Disregarding mere position, we are able to say, that if a curve $\phi(x, y) = 0$ is translated without rotation, so that a point of it (or indeed of its plane considered as moving with it) describes a circle whose radius is r, the envelope of the translated curve will be a parallel to the modulus r.

Thus if a curve be traced on the card of a mariner's compass, (the needle as usual being attached to the card,) and if the compass be moved with its centre on a circle and without oscillation, the envelope of the traced curve will be a parallel of it.

It will be observed that the name is now used to denote a curve which may be so placed as to become a parallel in the limited sense. When we are discussing the characteristics of the family of curves, mere position is immaterial.

3. It is instructive to consider a more general case. Let $\phi(x, y) = 0$, $\psi(x, y) = 0$ represent any two curves ϕ, ψ; then if ϕ be translated without rotation so that a point rigidly connected with it describes ψ, the envelope of the translated curve will be the same as if the offices of the curves were interchanged. For, in the first case, we have the system of curves $\phi(x-\alpha, y-\beta) = 0, \quad \psi(\alpha, \beta) = 0,$

and writing α' for $x-\alpha$, β' for $y-\beta$, we get the system

$$\psi(x-\alpha', y-\beta') = 0, \quad \phi(\alpha', \beta') = 0;$$

the envelope is the same whichever system be taken.

Translation without rotation implies that the curve translated continues to pass through the same points at infinity, and a system of parallel tangents remains similarly directed. The points at infinity of the envelope are (with higher multiplicity) the points at infinity of the two curves. Let m, n be the order and class of ϕ; m', n' those of ψ. If we consider the generation of the envelope by intersections, suppose,

of ϕ in successive positions as it is translated along a real branch of ψ, passing off to infinity, we have ultimately the curve ϕ sliding along its tangents parallel to the asymptote of the branch, while the distances between the points of contact of the tangents tend to vanish because all points at a finite distance from each other coalesce at infinity. The point at infinity in question becomes consequently a multiple point in the envelope of the order n. Hence the points where ψ meets the line at infinity are multiple points of the order n, and similarly the points where ϕ meets the line at infinity are multiple points of the order n', and the order of the envelope is in general $m'n+mn'$. The principle of continuity allows us to extend this conclusion to cases in which the points at infinity are imaginary.

4. As a particular case of the foregoing, the order of the parallel of ϕ is $2m+2n$; the points at infinity of ϕ are double points on the parallel, and the circular points are multiple points of the order n.

But from what precedes, we see at once that the line at infinity and the circular points have a special influence on the envelope. If the primitive touches the line at infinity, this line becomes part of the envelope, and the order is reduced. A reduction also takes place when the primitive passes through the circular points.

The parallel of a conic generally is of the 8th degree, that of a parabola is of the 6th. Hence the reduction is 2. And it may be readily inferred that for p simple contacts, the reduction of the order is $2p$. Again, the parallel of a circle is of the 4th degree, and the reduction is 4; and hence we may infer, generally, that for ordinary multiplicity of the order p at the circular points, the reduction is $4p$. Again, if there is a double point (or cusp) on the primitive, a circle, whose radius is the modulus described about the point as centre, satisfies the geometrical conditions.

The reduction for a double point is 4, and for a cusp 6, as appears from the expression for the order $2m+2n$.

(5.) From the form of the equations (a) it appears that if we make the first equation homogeneous in x, y, z, by introducing z for unity, z^2 and r^2 enter the equation of the envelope together; and, consequently, if we make $r^2=0$, the remaining terms will be of the same order as the general equation, *i.e.*, will contain the highest terms in x, y. The order of the residuum can be determined without much difficulty.

In the first place, we have, as factor, the square of the function of the primitive, for we have seen this is the function of the parallel when the modulus is evanescent. Secondly, we have the factors which represent the circular asymptote through each of the real ordinary foci. For if a, b are the coordinates of a focus, the normal distance from

$$(x-a)^2+(y-b)^2=0$$

satisfies the conditions, being evanescent. There are, in general, n real

ordinary foci, and the equation of the parallel is of the form

$$\chi_{2m+2n-2}\, r^2 + \{\phi(x, y)\}^2 \{(x-a_1)^2+(y-b_1)^2\} \ldots \{(x-a_n)^2+(y-b_n)\} = 0.$$

If the primitive has p simple contacts with the line at infinity, we have seen the reduction is $2p$. In this case, p foci pass off to infinity, and the part of the equation of the parallel independent of r^2 remains of the same form, there being, however, only $n-p$ focal factors.

Again, if the primitive passes simply through the circular points, and a, b are the coordinates of the corresponding double focus, the equation of the general parallel will be divisible by

$$\{(x-a)^2+(y-b)^2\}^2;$$

for the normal distance of the double focus may be taken as r, since for the circular points we have

$$(x-a)^2+(y-b)^2 = r^2.$$

The factor will be squared, because the focus is a double focus, *i.e.*, takes the place of two ordinary foci.

In like manner, if the primitive has μ foci due to tangents at the circular points and taking the place of ν ordinary foci, we have a reduction of the order of the parallel by 2ν. In this case also, if we make $r^2 = 0$ after removing the factors due to the foci formed by tangents at the circular points, the residuum remains of the same form containing the function of the primitive squared and the ordinary focal factors.

Thus it appears that the order of the parallel generally is $2m+2M$ where M is the number of real ordinary foci.

Suppose that the primitive passes through the circular points which are multiple points of the order p with s coincident tangents.

A multiple point of the order p with s coincident tangents is equivalent to $\frac{p(p-1)}{2} - s + 1$ double points, and $s-1$ cusps. The number of tangents which can be drawn from such a point is

$$m^2 - m - 2\left(\delta - \frac{p(p-1)}{2} + s - 1\right) - 3(\kappa - s + 1) - p(p+1),$$

δ, κ representing the numbers of the double points and cusps, or their equivalents on the primitive.

The above expression is, in fact,

$$n - 2p + s - 1.$$

The reduction of the order of the parallel is $2(2p-s+1)$.

Again, if the circular points are double points on the primitive, and the line at infinity is a common tangent, we can draw from each circular point $m^2 - m - 2(\delta - 1) - 3\kappa - 2 \cdot 3 - 1$ tangents. Hence the order of the parallel is

$$2m + 2n - 10.$$

If the circular points are cusps, and the line at infinity is the common tangent, we have, for the number of tangents which can be drawn, as before, $m^2-m-2\delta-3(\kappa-1)-2\,.\,3-1$,
and the order of the parallel is $2m+2n-8$.

6. Practically, when a primitive is given, we can ascertain the number and position of real double points and cusps, by determining the positions in which a circle whose radius is the modulus will doubly touch or osculate the curve. Thus we see that a parallel of an ellipse can be made to show two real double points and four real cusps. A parallel of a parabola can be drawn to show one real double point and two real cusps, and that of a hyperbola may show four real double points and four real cusps.

An easy way is suggested by the foregoing remark of determining the number of cusps on the parallel of a curve without singularities, and without special relation to the line at infinity. We have, in fact, simply the conditions [writing ϕ for $\phi(x, y)$],

$$\phi=0, \quad (n-1)^4\left\{\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2\right\}^3-r^2\mathrm{H}^2=0,$$

where H represents the Hessian. The latter equation is obtained immediately from the expression for the radius of curvature.

The number of intersections of these curves is

$$6m(m-1)=6n.$$

Knowing, then, the order $2n+2m$, the class $2n$, and the cusps $6n$, in this case, the other characteristics follow from Plücker's equations.

For such a primitive, and even for one more general, having contacts with the line at infinity, we may use a still more simple consideration.

There being no double points or cusps on the primitive, the only way in which points of inflexion can arise on the parallel will be by means of the points of inflexion on the primitive; and evidently to each such point will correspond two on the parallel. Hence the number of inflexions on the parallel is $2i$, i being the corresponding number for the primitive. And this reasoning probably holds good when the primitive passes simply through the circular points.

When the primitive has n real ordinary foci, the circular points are, as we have seen, multiple points of the order n on the parallel. There are consequently no real ordinary finite foci on the general parallel. This indeed appears to be true always for a general modulus, and might be anticipated from the manner in which focal factors enter the equation of a parallel.

When the primitive is compound, the parallel manifestly breaks up into the parallels of the compound curves. The parallel of the modulus s of a parallel to the modulus r consists of a compound of parallels. In fact, since parallels of a parallel are parallels, and the general

parallel is of determinate order, the result of repetitions of the generation must be compounded of parallels to various moduli.

The result of taking parallels of parallels, where certain simple relations exist between the moduli, can be readily traced. Thus a curve may become part parallel to itself for other than evanescent values of the modulus.

7. In order to obtain the general characteristics of a parallel, I shall now make use of the substitution $\frac{1}{\rho} + r$ for $\frac{1}{\rho}$ in the reciprocal of the primitive, which substitution gives the reciprocal of the parallel. (Salmon's Higher Plane Curves, p. 273.)

Let a point curve of the degree n be represented by the equation

$$(n)\,\rho^n + (n-1)\,\rho^{n-1} + \ldots + (2)\,\rho^2 + (1)\,\rho + (0) = 0 \ldots\ldots\ldots (b),$$

where ρ is the radius vector; (0), (1), (2) represent A, B $\cos\theta$ + C $\sin\theta$, D $\cos^2\theta$ + E $\cos\theta \sin\theta$ + F $\sin^2\theta$; and generally (s) stands for a homogeneous function of $\cos\theta$, $\sin\theta$ of the order s.

The substitution in question gives, after multiplication by ρ^n,

$$(n)\,\rho^n + (n-1)\,\rho^{n-1}(1+r\rho) + \ldots + (2)\,\rho^2(1+r\rho)^{n-2} + (1)\,\rho(1-r\rho)^{n-1} + (0)\,(1+r\rho)^n = 0 \ldots\ldots\ldots (c).$$

If $\chi(x, y, 1)$ is the equivalent of (b), in x, y coordinates, the above result is

$$\chi(x, y, 1+r\rho) = 0 \ldots\ldots\ldots (d).$$

And since the value of ρ is radical, we have also

$$\chi(x, y, 1-r\rho) = 0 \ldots\ldots\ldots (e).$$

The product of (d), (e) is the reciprocal of the parallel of the reciprocal of (b), or its tangential equation, if we choose so to regard it.

Writing χ for $\chi(x\ y, z)$ where z represents unity, we have the form

$$\left\{\chi + \frac{d^2\chi}{dz^2}\frac{r^2\rho^2}{1.2} + \frac{d^4\chi}{dz^4}\frac{r^4\rho^4}{1.2.3.4} + \ldots\right\}^2 - r^2\rho^2\left\{\frac{d\chi}{dz} + \frac{d^3\chi}{dz^3}\frac{r^2\rho^2}{1.2.3} + \ldots\right\}^2 = 0 \ldots\ldots (f).$$

As long as ρ^2 does not vanish for a singular point,—and in the general case we may make this supposition,—a double point on either of the part curves (d), (e) implies a corresponding singular point of the same nature on the primitive. If, therefore, the curve (b) has no singularities, the singularities of (f) will consist of the intersections of

$$\chi + \frac{d^2\chi}{dz^2}r^2\rho^2 + \ldots = 0, \quad \text{and} \quad \frac{d\chi}{dz} + \frac{d^3\chi}{dz^3}\frac{r^2\rho^2}{1.2.3} + \ldots = 0.$$

The curve (f) is in this case of the degree $2n$, and has $n(n-1)$ double points. We are, therefore, in a position to determine the characteristics of the parallel of a curve of the degree $n(n-1)$ and class n, having no special relation to the line at infinity.

8. Let us now suppose that the curve (b) has α double points and β cusps, we shall have in general 2α double points and 2β cusps in the

transformed curve; for to each double point or cusp x_1, y_1, in (b), correspond two double points or cusps $\frac{x}{1\pm r\rho} = x_1$, $\frac{y}{1\pm r\rho} = y_1$, in (f).

It will be seen presently that, in special cases, there may be a modification of the characteristics, which makes it convenient to add ϵ to the number of double points, and η to the number of cusps in the general formulæ. For $2\epsilon+3\eta$ I write Q, and

M for the order,
N „ class,
D „ number of double points,
C „ „ „ cusps,
I „ „ stationary tangents,
T „ „ double tangents.

It is to be understood, however, that we must expect only the equivalents of these elements. For instance, generally, the circular points at infinity are multiple points of this order $\frac{M-2q}{2}$, where q is the number of points, not circular points, at infinity on the primitive. It would, I think, be convenient to adopt a less definite phrase than "number of double points," &c., such as "nodal characteristic," "cuspidal characteristic," &c. I shall also call $\delta+\kappa$ the *multiplicity*, and $2\delta+3\kappa$ the *distribution*. This phraseology may be used with reference to an isolated singularity, or to the sum total of the singularities, of a curve. In the latter case, we have

$$\frac{(m-1)(m-2)}{2} - \text{multiplicity} = \text{deficiency},$$

a term introduced, I think, by Prof. Cayley.

Let the primitive have δ double points, and κ cusps, then for the parallel we have

$$\begin{aligned}
N &= 2n,* \\
I &= 6n-6m+2\kappa+\eta, \\
T &= (n-m)(n+m-9)+2\delta+n(n-1)+\epsilon, \\
M &= 2m+2n-Q, \\
C &= 6n+2\kappa-3Q+\eta, \\
D &= 2(n+n)^2-11(n+n)+10m-3\kappa \\
&\quad +\frac{Q^2}{2}-Q\left(2m+2n-\frac{9}{2}\right)+\epsilon.
\end{aligned}$$

If i, t are the numbers of stationary tangents and double tangents on the primitive, we have $I = 2i+\eta$,

$$T = 2t+n(n-1)+\epsilon.$$

* The value of the class follows at once from the consideration, that for each of any set of parallel tangents to the primitive we must have two tangents to the parallel similarly directed.

When the primitive has no special relation to the line at infinity,

$$\eta = \epsilon = Q = 0.$$

It appears, then, that I remains equal to $2i$ when there are singular points on the primitive; but this is not so manifest, *à priori*, as to give the characteristic at once.*

9. We have now to consider cases in which a singularity of the form (f) satisfies $\rho^2 = 0$. These cases relate to passages through the circular points and contact with the line at infinity.

Let u_s, w_s, p_s represent homogenous functions of x, y of the order s. The form $\{u_n+u_{n-1}+ \dots +u_p\}^2-p_1 q_1 \{w_{n-1}+w_{n-2}+ \dots w_{p-1}\} = 0$ represents a curve, having at the origin a multiple point of the order $2p$, and $n(n-1)-p(p-1)$ other double points. Removing w_{p-1} altogether, we make p of the double points adjacent to the origin, which remains a multiple point of the order $2p$. If, again, we make w_p identical with u_p, except as to a constant factor, we bring another set of p double points into contiguity with the origin. We have, therefore, the equivalent of

$$n(n-1)-p(p-1)+\frac{2p(2p-1)}{2} = n(n-1)+p(p-1)+p$$

double points. Or we may get the same result thus:—If, in the form

$$(u_n+u_{n-1}+ \dots u_p)^2-p_1 q_1 (w_{n-1}+ \dots +w_{p+1}+u_p)^2 = 0 \quad \dots\dots\dots (g),$$

we suppose p_1, q_1 to coincide, we have the equivalent of n^2 intersections of two component curves of the order n; and we add $n-p$ points to the multiplicity, since p_1, q_1 pass through the origin which is of the order p on $u_n+ \dots = 0$, $w_{n-1}+ \dots = 0$.

Let Δ be the multiplicity of the form (g), then we have

$$\Delta + n-p = n^2-p^2+\frac{2p(2p-1)}{2},$$

$$\Delta = n(n-1)+p(p-1)+p.$$

10. If the primitive has p simple contacts with the line at infinity, the origin is a multiple point of the order p on the reciprocal. The transformed curve becomes of the form

$$(u_n+u_{n-1}+ \dots +u_p)^2-r^2\rho^2 (w_{n-1}+ \dots +u_p)^2 = 0 \dots\dots\dots (g').$$

Hence there are due to the tangency $p(p-1)+p$ double points, of which $p(p-1)$ are due to the multiple tangent as such, and the re-

* Since the present paper was read before the Society, I have had the advantage of comparing my formulæ with those of Professor Cayley, given in the sequel of his paper on Evolutes and Parallel Curves (April number of the "Quarterly Journal of Mathematics"). Our corresponding results seem to be generally accordant. As to the point referred to in the text, Professor Cayley had concluded that $I = 2i$ in all cases. On my drawing his attention to the case of the tricusped hypocycloid, the parallel of which does not break up as it ought if the above equation held good, he recognised that the assumption is too general. It follows that the parallels of curves having no foci do not necessarily break up.

mainder p are due to the position of the tangent. Hence, in the formulæ of (8), we must write $\eta = 0$, $\epsilon = p$, $Q = 2p$.

If some of the contacts coalesce, the effect of the tangency on the characteristics remains the same.

For suppose u_p is of the form $v_1^s u_{p-s}$. We have at the origin, in each of the intersecting curves $u_n + \ldots = 0$, $w_{n-1} + \ldots = 0$, the equivalent of $(s-1)$ cusps, and on the curve (g') the equivalent of $2(s-1)$ cusps, but these are due to the multiple tangent as such, so that the addition on account of its position is unaltered.

The fact that multiple points of the primitive lie at infinity does not of itself alter the characteristics. But if a double point is at infinity, and one of its tangents coincides with the line at infinity, the origin will be a point on the reciprocal curve. We shall have to apply the same reasoning as in the case of a simple contact with the line at infinity. The reduction of the order will be 2; and in like manner the reduction on account of p such tangencies will be $2p$. If there is a multiple point at infinity of the order p, with s tangents altogether at infinity, the reducing effect of the tangency is separate and equal to s simple tangencies.

11. Let the form

$$(xy\phi_{n-2} + \psi_p^2 \chi_{n-2p})^2 + xy\{xy\xi_{n-3} + \psi_p \zeta_{n-p-1}\}^2 = 0 \quad \ldots\ldots\ldots\ldots (h),$$

or

$$\Phi^2 + xy\Psi^2 = 0,$$

be homogeneous in x, y, z.

For double points we have

$$2\Phi\frac{d\Phi}{dx} + y\Psi^2 + 2xy\Psi\frac{d\Psi}{dx} = 0,$$

$$2\Phi\frac{d\Phi}{dy} + x\Psi^2 + 2xy\Psi\frac{d\Psi}{dy} = 0,$$

$$2\Phi\frac{d\Phi}{dz} + 2xy\Psi\frac{d\Psi}{dz} = 0.$$

We have, therefore, upon $xy = 0$, the double points

$$x = 0, \quad \psi_p^2 = 0,$$
$$y = 0, \quad \psi_p^2 = 0.$$

On the face of the form (h) and independently of the factor xy, we have $n(n-1)$ double points, including the systems

$$\left.\begin{aligned} x &= 0, \quad \psi_p = 0 \\ y &= 0, \quad \psi_p = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (k).$$

Consequently, by virtue of the factor xy, we have p double points on $x=0$ and p double points on $y=0$ adjacent to the double points (k).

12. If the circular points at infinity are multiple points of the order p on the primitive, we have the circular asymptotes through the origin multiple tangents to the reciprocal touching at p points.

For simplicity, let $p=2$, $n=4$; the reciprocal of the primitive will be of the form $\rho^3(u_2+u_1+u_0)+(w_2+w_1+w_0)^2=0$,

and the transformed equation is of the form

$$\{\rho^2\chi_2+(w_2+w_1+w_0)^2\}^2+r^2\rho^2\{\rho^2\xi_1+2(w_1+2w_0)(w_2+w_1+w_0)\}^2=0.$$

By the preceding article, we have two additional double points on $x+\sqrt{-1}\,y=0$ and two on $x-\sqrt{-1}\,y=0$; so that, in the formulæ of (8), we have $\eta=0$, $\epsilon=4$, $Q=8$. Generally, for p passages through the circular points, we have $\eta=0$, $\epsilon=2p$, $Q=4p$.

13. Let the circular points be cusps on the primitive. For simplicity, let $n=3$; the reciprocal may be written in the form

$$\rho^2\chi_1+Z^3=0,$$

and the transformed equation is

$$(\rho^2\xi_1+Z^3)^2-r^2\rho^2(k\rho^2+lZ^2)^2=0.$$

We have, for double points or cusps on $\rho^2=0$,

$$\rho^2=0,\quad Z^3=0;$$

but the form gives on the face of it 3.2 double points including the double points $\rho^2=0$, $Z^2=0$.

We have, therefore, for the multiplicity of the addition,

$$\Delta+K=2.$$

But we know, by (5), that the distribution is

$$2\Delta+3K=6.$$

Hence $K=2$, $\Delta=0$.

In the formulæ of (8), we must write $\eta=2$, $\epsilon=0$, $Q=6$. We may generalize this equation by means of a compound primitive.

14. If the primitive touches the line at infinity at the circular points, the origin will be a double point on the reciprocal, and the circular asymptotes through the origin will be its tangents. For to the line at infinity corresponds the origin, and to the circular points correspond the circular asymptotes through the origin. Hence we have to consider the form $(u_n+u_{n-1}+\ldots+\rho^2)^2-r^2\rho^2(w_{n-1}+\ldots+\rho^2)^2=0$.

This is a particular case of the form (g') where $p=2$, restricted by the coincidence of u_2 with ρ^2.

The multiplicity of the addition is 2, or

$$\Delta+K=2.$$

But we know, by (5), that the distribution is $2\Delta+3K=6$,

so that $K=2$, $\Delta=0$.

We must write, in the formulæ of (8), $\eta=2$, $\epsilon=0$, $Q=6$.

The difficulty of the subject rapidly increases as the relation of the primitive to the circular points becomes complex. These points, for instance, may be cusps with a common tangent altogether at infinity.

&c. I have not been able to satisfy myself as to the effect of such singularities generally.*

15. As a partial verification of the formulæ of (8), we will consider the parallel of the parallel of a primitive with δ double points and κ cusps, and having no special relation to the line at infinity.

We have seen that, the circular points being multiple points of the order n on the first parallel, give rise to $2n$ additional double points on the reciprocal of the second parallel. Making $\eta=0$, $\epsilon=2n$, $Q=4n$, and otherwise appropriately substituting in the formulæ of (8), we have

$$M = 4(m+n),$$
$$N = 4n,$$
$$D = 4(m+n)^2+2\{2(m+n)^2-11(m+n)+10m-3\kappa\}.$$

The term $4(m+n)^2$ in the last expression is due to the intersections of the two parallels of which the second parallel is compounded. The numbers of the other characteristics are, as they should be, the numbers of the characteristics of the first parallel doubled.

16. I give some results relative to the parallels of a few of the simpler curves :—

The parallel of a conic, generally, is of the 8th degree, with 8 double points and 12 cusps.

The parallel of a parabola is of the 6th degree, with 4 double points and 6 cusps. It is unicursal.

The parallel of a tricusped circular quartic is of the degree 10, with 24 double points and 12 cusps. It is unicursal.

The parallel of a cardioid is of the degree 8, with 13 double points and 8 cusps. It is unicursal.

The parallel of a bicircular quartic with a cusp is of the degree 10, with 28 double points and 8 cusps. It is unicursal.

The parallel of a tricusped hypocycloid is of the degree 8, with 13 double points and 8 cusps. It is unicursal.

In tracing parallels, it is well to observe that the curve will not always exhibit its full degree. Manifestly, a small modulus makes the form of the parallel approach to that of the primitive doubled. Attention must be paid to this, if it is desired to exhibit all the real singularities of which a parallel is susceptible.

Whilst giving his account of the methods employed in the above paper, Mr. Roberts expressed his desire to record a construction which he had omitted to see at the proper moment, though at different times he had noted two conclusions which, when brought together, make it obvious enough. Stated as a theorem, it is as follows :—In a plane, if

* The difficulty arises from the circumstance that $I=2i$ does not necessarily hold in these cases. Otherwise, knowing the order, the class, and the number of inflexions, we should have the other singularities by Plücker's equations. Even in the simple cases of §§ 13, 14, I give the results with diffidence.

a limited straight line whose length is equal to the distance between the centres of two equal circles, moves with an extremity on each, the locus of any point rigidly connected with the line will consist of a circle and a bicircular quartic with a third node.

This construction is mechanically more convenient than the one discussed in Mr. Roberts' paper "On the Pedals of Conics," and the two constructions are intimately connected with each other, and belong to the same theory.

Prof. Peirce, of Harvard University, laid before the Society some details of the methods made use of by him in his work on "Linear Associative Algebra."

The President conveyed to the Author the thanks of the Society for his interesting communication; and, at the request of Dr. Hirst, Prof. Peirce presented a copy of his work to the Library of the Society.

The following presents were received:—

"Linear Associative Algebra," by Benjamin Peirce, LL.D., Perkins Professor of Mathematics and Astronomy in Harvard University: from the author.

"Transactions of the Connecticut Academy of Arts and Sciences," vol. I., part 1; vol. II., part 1: from the Connecticut Academy.

"Nautical Almanac for 1874:" from the Nautical Almanac Office.

"Monatsbericht" for August, September, and October, 1870.

"Crelle," 72 Band, 4es Heft.

Feb. 9th, 1871.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Mr. C. R. Hodgson, B.A. Lond., was proposed for election, and the Rev. J. Wolstenholme, M.A., and Mr. R. B. Hayward, M.A., were elected Members.

Prof. Cayley made a communication

On an Analytical Theorem from a New Point of View.

The theorem is a well known one, derived from the equation

$$(az^2+2bz+c)\,w^2+2\,(a'z^2+2b'z+c')\,w+a''z^2+2b''z+c''=0;$$

viz., considering this equation as establishing a relation between the variables z and w, and writing it in the forms

$$2u=\mathrm{A}w^2+2\mathrm{B}w+\mathrm{C}=\mathrm{A}'z^2+2\mathrm{B}'z+\mathrm{C}'=0,$$

(where, of course, A, B, C are quadric functions of z, and A′, B′, C′ quadric functions of w,) we have

$$0=\frac{du}{dw}\,dw+\frac{du}{dz}\,dz=(\mathrm{A}w+\mathrm{B})\,dw+(\mathrm{A}'z+\mathrm{B}')\,dz;$$

but in virtue of the equation $u = 0$, we have $Aw+B = \sqrt{B^2-AC}$, and $A'z+B' = \sqrt{B'^2-A'C'}$, and the differential equation thus becomes

$$\frac{dw}{\sqrt{B'^2-A'C'}} + \frac{dz}{\sqrt{B^2-AC}} = 0,$$

where $B'^2-A'C'$ and B^2-AC are quartic functions of w and z respectively. This is, of course, integrable (viz., the integral is the original equation $u=0$); and it follows, from the theory of elliptic functions, that the two quartic functions must be linearly transformable into each other; viz., they must have the same absolute invariant $I^3 \div J^2$. It is, in fact, easy to verify, not only that this is so, but that the two functions have the same quadrinvariant I, and the same cubinvariant J.

The new point of view is, that we take the coefficients a, b, &c., to be homogeneous functions of (x, y), their degrees being such that the equation $u=0$ is a quartic equation $(*)(x, y, z, w)^4 = 0$; viz., this equation now represents a quartic surface having a node (conical point) at the point $(x=0,\ y=0,\ z=0)$, and also a node at the point $(x=0,\ y=0,\ w=0)$, say, these points are O, O′ respectively. The equation $B'^2-A'C = 0$ gives the circumscribed sextic cone having O for its vertex, and the equation $B^2-AC = 0$ the circumscribed sextic cone having O′ for its vertex; each of these cones has the line OO′ $(x=0,\ y=0)$ for a nodal line, as appears geometrically, and also by the equations containing z, w respectively in the degree 4. Considering $B'^2-A'C'$ as a quartic function of z, its quadrinvariant is a function $(x, y)^8$, and its cubinvariant a function $(x, y)^{12}$; and similarly, considering B^2-AC as a quartic function of w, its invariants are functions $(x, y)^8$ and $(x, y)^{12}$. We have thus, between the two cones, a geometrical relation answering to the analytical one of the identity of the invariants; but the nature of this geometrical relation is not obvious; and it presents itself as an interesting subject of investigation.

Prof. Cayley made also a communication

On a Problem in the Calculus of Variations.

The problem is, $z = \frac{1}{3}(3x-y^2)\,y$, to find y a function of x such that $\int z dx = \text{max. or min.}$, subject to a given condition $\int y dx = c$ (the limits of each integral being x_1, x_0, where these quantities are each positive, and $x_1 > x_0$). The ordinary method of solution gives $y^2 = x+\lambda$, where $(x_1+\lambda)^{\frac{3}{2}} - (x_0+\lambda)^{\frac{3}{2}} = \frac{3}{2}c$; so long as c is not less than $(x_1-x_0)^{\frac{3}{2}}$, there is a real value of λ, but for a smaller value of c there is no real value. The difficulty arising in this last case is somewhat illustrated by replacing the original problem by a like problem of ordinary maxima and minima; viz., $x_1, x_2 \ldots x_n$ being given positive values of x, in the order of increasing magnitude; and if, in general, $z_i = (3x_i-y_i^2)\,y_i$, then

the problem is to find y_i a function of x_i, such that $\Sigma z_i = \text{max. or min.}$, subject to the condition $\Sigma y_i = c$. We have here $y_i^2 = x_i + \lambda$, where λ is to be determined by the condition $\Sigma y_i = c$; the remainder of the investigation turns on the question of the sign $y_i = +\sqrt{x_i+\lambda}$ or $y_i = -\sqrt{x_i+\lambda}$, to be taken for the several values of i respectively.

Prof. Henrici exhibited a plaster model of a Tubular Surface of the 6th order, for the generation of which the two following modes may be employed. Either a sphere of constant radius moves with its centre on a parabola, or it rolls along the same parabola, always touching both its branches. The two envelopes thus produced differ in position only. The second mode of generation shows that the surface has a nodal curve, which is a parabola congruent to that on which the centre of the sphere moves, but in a plane perpendicular to it. Through a part of it only real sheets of the surface pass. There is also a cuspidal curve of the 6th order which has two cusps. The nodal curve passes through them, and has, at the cusps, the same tangents. The equation to the surface is

$$(27py^2+9x\mathrm{K}-x^3)^2 = (x^2+3\mathrm{K})^3,$$

where

$$\mathrm{K} = (x+2p)^2+y^2+z^2-r^2,$$

and r is the radius of the sphere and $4p$ the parameter of the parabola. The equations to the parabola on which the centre of the sphere moves are

$$y^2 = 4p\,(x+2p), \quad z = 0;$$

those to the nodal curve

$$y = 0, \quad z^2 = -4px+r^2-4p^2;$$

whilst those to the cuspidal curve are

$$27py^2-4x^3 = 0, \quad x^2+3\mathrm{K} = 0,$$

the first denoting a cylinder which cuts the plane $z = 0$ in the evolute of the parabola, and the second representing an ellipsoid of revolution.

The model was constructed to the scale $p = \frac{1}{20}$ inch, $r = 2$ inches.

At the suggestion of the Treasurer, Prof. Henrici was requested, by the members present, to order a second model to be cast for the Society's collection.

Mr. Merrifield, F.R.S., communicated the following property of Conical and Cylindrical Surfaces, which he thought to be new :—

If the equation of a surface be

$$z = \mathrm{F}(x, y) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

it is very well known that the condition that it should be a ruled surface is that

$$\left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}\right)^2 z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and

$$\left(\lambda\frac{d}{dx}+\mu\frac{d}{dy}\right)^3 z \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

should have a common factor of the form $\mathrm{A}\lambda+\mathrm{B}\mu$, and also that the condition of its being developable is that (2) should have two equal factors of that form.

I have found, upon actual trial, that for a conical surface (3) will have two equal factors, and for a cylindrical surface three equal factors, that is to say, if we write $\alpha = \frac{d^3z}{dx^3}$, $\beta = \frac{d^3z}{dx^2dy}$, &c., we have for a conical surface $(\alpha\delta-\beta\gamma)^2 = (\alpha\gamma-\beta^2)(\beta\delta-\gamma^2)$,

and for a cylindrical surface we have separately

$$\alpha\delta-\beta\gamma = 0, \quad \alpha\gamma-\beta^2 = 0, \quad \beta\delta-\gamma^2 = 0.$$

In fact, if we use the equation of a cone

$$z-c = (x-a)\,\mathrm{F}\left(\frac{y-b}{x-a}\right),$$

we find, by differentiation,

$$\alpha\gamma-\beta^2 = -\frac{(y-b)^2}{(x-a)^6}\mathrm{F}''^2,$$

$$\beta\delta-\gamma^2 = -\frac{1}{(x-a)^4}\mathrm{F}''^2,$$

$$\alpha\delta-\beta\gamma = \frac{2\,(y-b)}{(x-a)^5}\mathrm{F}''^2,$$

whence $(\alpha\delta-\beta\gamma)^2 = 4\,(\alpha\gamma-\beta^2)(\beta\delta-\gamma^2)$.

If we use the equation of a cylinder $ny-mz = \mathrm{F}\,(mx-ly)$,

we obtain

$$m\alpha = -m^3\mathrm{F}''',$$
$$m\beta = \quad lm^2\mathrm{F}''',$$
$$m\gamma = -l^2m\mathrm{F}''',$$
$$m\delta = \quad l^3\mathrm{F}''',$$

whence $\left(\lambda\frac{d}{dz}+\mu\frac{d}{dy}\right)^3 z$ takes the form

$$\frac{\mathrm{F}'''}{m}(l\mu-m\lambda)^3.$$

I have not yet had time to look into the questiou, whether the converse of the proposition is true, namely, whether the introduction of the condition of developability ($rt = s^2$) necessarily reduces the surface, in which two or three of the roots of (3) are equal, to a cone or cylinder.

Dr. Hirst made some remarks on the connection between the correlation of two planes, as described in his late communication to the Society, and Sturm's solution of the problem of Projectivity, as given by him in his memoir on the subject published in the "Mathematische Annalen," vol. I., p. 533.

The following presents were received :—

Nos. 1, 2, 3 of the Journal of the London Institution.

"On Hills and Dales," by J. Clerk Maxwell, F.R.S.; and an Address to the Mathematical and Physical Section of the British Association, Sept. 15th, 1870, by the same: both from the Author.

The Annual of the Royal School of Naval Architecture and Marine Engineering: from Mr. Merrifield (the Principal).

March 9th, 1871.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Mr. C. R. Hodgson, B.A. Lond., was elected a Member; and the following gentlemen were proposed for election:—The Hon. J. W. Strutt; Major Frederick Close, R.A.; and Mr. James Stuart, Fellow and Tutor of Trinity College, Cambridge.

Two Models were exhibited by Professor Henrici, the one being the cast ordered for the Society at the February meeting, the other a model of the surface described at the meeting held November, 1869; it was agreed that Professor Henrici should order a copy of the latter model also to be made for the Society.

Professor H. J. S. Smith read a paper on "Skew Cubics," upon which Professor Cayley made a few remarks.

Mr. Tucker (Hon. Sec.) then read the following communication:—

Remarks on the Mathematical Classification of Physical Quantities, by J. Clerk-Maxwell, LL.D., F.R.S.

The first part of the growth of a physical science consists in the discovery of a system of quantities on which its phenomena may be conceived to depend. The next stage is the discovery of the mathematical form of the relations between these quantities. After this, the science may be treated as a mathematical science, and the verification of the laws is effected by a theoretical investigation of the conditions under which certain quantities can be most accurately measured, followed by an experimental realisation of these conditions, and actual measurement of the quantities.

It is only through the progress of science in recent times that we have become acquainted with so large a number of physical quantities that a classification of them is desirable.

One very obvious classification of quantities is founded on that of the sciences in which they occur. Thus temperature, pressure, density, specific heat, latent heat, &c., are quantities occurring in the theory of the action of heat on bodies.

But the classification which I now refer to is founded on the mathematical or formal analogy of the different quantities, and not on the matter to which they belong. Thus a finite straight line, a force, a velocity of rotation, &c., are quantities, differing in their physical nature, but agreeing in their mathematical form. We may distinguish the two methods of classification by calling the first a physical, and the second a mathematical classification of quantities.

A knowledge of the mathematical classification of quantities is of

great use both to the original investigator and to the ordinary student of the science. The most obvious case is that in which we learn that a certain system of quantities in a new science stand to one another in the same mathematical relations as a certain other system in an old science, which has already been reduced to a mathematical form, and its problems solved by mathematicians.

Thus, when Mossotti observed that certain quantities relating to electro-static induction in dielectrics had been shown by Faraday to be analogous to certain quantities relating to magnetic induction in iron and other bodies, he was enabled to make use of the mathematical investigation of Poisson relative to magnetic induction, merely translating it from the magnetic language into the electric, and from French into Italian.

Another example, by no means so obvious, is that which was originally pointed out by Sir William Thomson, of the analogy between problems in attractions and problems in the steady conduction of heat, by the use of which we are able to make use of many of the results of Fourier for heat in explaining electrical distributions, and of all the results of Poisson in electricity in explaining problems in heat.

But it is evident that all analogies of this kind depend on principles of a more fundamental nature; and that, if we had a true mathematical classification of quantities, we should be able at once to detect the analogy between any system of quantities presented to us and other systems of quantities in known sciences, so that we should lose no time in availing ourselves of the mathematical labours of those who had already solved problems essentially the same.

All quantities may be classed together in one respect, that they may be defined by means of two factors, the first of which is a numerical quantity, and the second is a standard quantity of the same kind with that to be defined.

Thus number may be said to rule the whole world of quantity, and the four rules of arithmetic may be regarded as the complete equipment of the mathematician.

Position and form, which were formerly supposed to be in the exclusive possession of geometers, were reduced by Descartes to submit to the rules of arithmetic by means of that ingenious scaffolding of coordinate axes which he made the basis of his operations.

Since this great step was taken in mathematics, all quantities have been treated in the same way, and presented to the mind by means of numbers, or symbols which denote numbers, so that as soon as any science has been thoroughly reduced to the mathematical form, the solution of problems in that science, as a mental process, is supposed (at least by the outer world) to be carried on without the aid of any of the physical ideas of the science.

I need not say that this is not true, and that mathematicians, in solving physical problems, are very much aided by a knowledge of the science in which the problems occur.

At the same time, I think that the progress of science, both in the way of discovery, and in the way of diffusion, would be greatly aided if more attention were paid in a direct way to the classification of quantities.

A most important distinction was drawn by Hamilton when he divided the quantities with which he had to do into Scalar quantities, which are completely represented by one numerical quantity, and Vectors, which require three numerical quantities to define them.

The invention of the calculus of Quaternions is a step towards the knowledge of quantities related to space which can only be compared, for its importance, with the invention of triple coordinates by Descartes. The ideas of this calculus, as distinguished from its operations and symbols, are fitted to be of the greatest use in all parts of science.

We may imagine another step in the advancement of science to be the invention of a method, equally appropriate, of conceiving dynamical quantities. As our conceptions of physical science are rendered more vivid by substituting for the mere numerical ideas of Cartesian mathematics the geometrical ideas of Hamiltonian mathematics, so in the higher sciences the ideas might receive a still higher development if they could be expressed in language as appropriate to dynamics as Hamilton's is to geometry.

Another advantage of such a classification is, that it guides us in the use of the four rules of arithmetic. We know that we must not apply the rules of addition or subtraction unless the quantities are of the same kind. In certain cases we may multiply or divide one quantity by another, but in other cases the result of the process is of no intellectual value.

It has been pointed out by Professor Rankine that the physical quantity called Energy or Work can be conceived as the product of two factors in many different ways.

The dimensions of this quantity are $\frac{ML^2}{T^2}$, where L, M, and T represent the concrete units of length, time, and mass. If we divide the energy into two factors, one of which contains L^2, both factors will be scalars. If, on the other hand, both factors contain L, they will be both vectors. The energy itself is always a scalar quantity.

Thus, if we take the mass and the square of the velocity as the factors, as is done in the ordinary definitions of vis-viva or kinetic energy, both factors are scalar, though one of them, the square of the velocity, has no distinct physical meaning.

Another division into apparently scalar factors is that into volume and hydrostatic pressure, though here we must consider the volume, not in itself, but as a quantity subject to increase and diminution, and this change of volume can only occur at the surface, and is due to a variation of the surface in the direction of the normal, so that it is not a scalar but a vector quantity. The pressure also, though the abstract conception of a hydrostatic pressure is scalar, must be conceived as applied at a surface, and thus becomes a directed quantity or vector.

The division of the energy into vector factors affords results always capable of satisfactory interpretation. Of the two factors, one is conceived as a tendency towards a certain change, and the other as that change itself.

Thus the elementary definition of Work regards it as the product of a force into the distance through which its point of application moves, resolved in the direction of the force. In the language of Quaternions, it is the scalar part of the product of the force and the displacement.

These two vectors, the force and the displacement, may be regarded as types of many other pairs of vectors, the products of which have for their scalar part some form of energy.

Thus, instead of dividing kinetic energy into the factors "mass" and "square of velocity," the latter of which has no meaning, we may divide it into "momentum" and "velocity," two vectors which, in the dynamics of a particle, are in the same direction, but, in generalized dynamics, may be in different directions, so that in taking their product we must remember the rule for finding the scalar part of it.

But it is when we have to deal with continuous bodies, and quantities distributed in space, that the general principle of the division of energy into two factors is most clearly seen.

When we regard energy as residing intrinsically in a body, we may measure its intensity by the amount contained in unit of volume. This is, of course, a scalar quantity.

Of the factors which compose it, one is referred to unit of length, and the other to unit of area. This gives what I regard as a very important distinction among vector quantities.

Vectors which are referred to unit of length I shall call Forces, using the word in a somewhat generalized sense, as we shall see. The operation of taking the integral of the resolved part of a force in the direction of a line for every element of that line, has always a physical meaning. In certain cases the result of the integration is independent of the path of the line between its extremities. The result is then called a Potential.

Vectors which are referred to unit of area I shall call Fluxes. The operation of taking the integral of the resolved part of a flux perpendicular to a surface for every element of the surface has always a

physical meaning. In certain cases the result of the integration over a closed surface is independent, within certain restrictions, of the position of the surface. The result then expresses the Quantity of some kind of matter, either existing within the surface, or flowing out of it, according to the physical nature of the flux.

In many physical cases, the force and the flux are always in the same direction, and proportional to each other. The one is therefore used as the measure of the other, their symbols degenerate into one, and their ideas become confounded together. One of the most important mathematical results of the discovery of substances having different physical properties in different directions, has been to enable us to distinguish between the force and the flux, by letting us see that their directions may be different.

Thus, in the ordinary theory of fluids, in which the only motion considered is that which we can directly perceive, we may define the velocity equally well in two different ways. We may define it with reference to unit of length, as the number of such units described by a particle in unit of time; or we may define it with reference to unit of area, as the volume of the fluid which passes through unit of area in unit of time. If defined in the first way, it belongs to the category of forces; if defined in the second way, to the category of fluxes.

But if we endeavour to develope a more complete theory of fluids, which shall take into account the facts of diffusion, where one fluid has a different velocity from another in the same place; or if we accept the doctrine, that the molecules of a fluid, in virtue of the heat of the substance, are in a state of agitation; then, though we may give a definition of the velocity of a single molecule with reference to unit of length, we cannot do so for the fluid; and the only way we have of defining the motion of the fluid is by considering it as a flux, and measuring it by the mass of the fluid which flows through unit of area.

This distinction is still more necessary when we come to heat and electricity. The flux of heat or of electricity cannot be even thought of in any way except as the quantity which flows through a given area in a given time. To form a conception of the velocity, properly so called, of either agent, would require us to conceive heat or electricity as a continuous substance having a known density.

We must therefore consider these quantities as fluxes. The forces corresponding to them are, in the case of heat, the rate of variation of temperature, and, in the case of electricity, the rate of variation of potential.

I have said enough to point out the distinction between forces and fluxes. In statical electricity the resultant force at a point is the rate of variation of potential, and the flux is a quantity, hitherto confounded with the force, which I have called the electric displacement.

In magnetism the resultant force is also the rate of variation of the potential, and the flux is what Faraday calls the magnetic induction, and is measured, as Thomson has shown, by the force on a unit pole placed in a narrow crevasse cut perpendicular to the direction of magnetization of the magnet. I shall not detain the Society with the explanation of these quantities, but I must briefly state the nature of a ratio of a force to a flux in its most general form.

When one vector is a function of another vector, the ratio of the first to the second is, in general, a quaternion which is a function of the second vector.

When, if the second vector varies in magnitude only, the first is always proportional to it, and remains constant in direction, we have the important case of the function being a *linear* one. The first vector is then said to be a linear and vector function of the second. If α, β, γ are the Cartesian components of the first vector, and a, b, c those of the second, then

$$\begin{aligned} \alpha &= r_1 a + q_3 b + p_2 c, \\ \beta &= p_3 a + r_2 b + q_1 c, \\ \gamma &= q_2 a + p_1 b + r_3 c, \end{aligned}$$

where the coefficients p, q, r are constants. When the p's are equal to the corresponding q's, the function is said to be self-conjugate. It may then be represented geometrically as the relation between the radius vector from the centre of an ellipsoid, and the perpendicular on the tangent plane.

We may remark that even here, where we may seem to have reached a purer air, uncontaminated by physical applications, one vector is essentially a line, while the other is defined as the normal to a plane, as in all the other pairs of vectors already mentioned.*

Another distinction among physical vectors is founded on a different principle, and divides them into those which are defined with reference to translation and those which are defined with reference to rotation. The remarkable analogies between these two classes of vectors is well pointed out by Poinsôt in his treatise on the motion of a rigid body. But the most remarkable illustration of them is derived from the two different ways in which it is possible to contemplate the relation between electricity and magnetism.

Helmholtz, in his great paper on Vortex Motion, has shown how to construct an analogy between electro-magnetic and hydro-kinetic phenomena, in which magnetic force is represented by the velocity of the fluid, a species of translation, while electric current is represented by the rotation of the elements of the fluid. He does not propose this as an explanation of electro-magnetism; for though the analogy is perfect in form, the dynamics of the two systems are different.

* The subject of linear equations in quaternions has been developed by Professor Tait, in several communications to the Royal Society of Edinburgh.

According to Ampère and all his followers, however, electric currents are regarded as a species of translation, and magnetic force as depending on rotation. I am constrained to agree with this view, because the electric current is associated with electrolysis, and other undoubted instances of translation, while magnetism is associated with the rotation of the plane of polarization of light, which, as Thomson has shown, involves actual motion of rotation.

The Hamiltonian operator $V\nabla$, applied to any vector function, converts it from translation to rotation, or from rotation to translation, according to the kind of vector to which it is applied.

I shall conclude by proposing for the consideration of mathematicians certain phrases to express the results of the Hamiltonian operator Δ. I should be greatly obliged to anyone who can give me suggestions on this subject, as I feel that the onomastic power is very faint in me, and that it can be successfully exercised only in societies.

∇ is the operation $$i\frac{d}{dx}+j\frac{d}{dy}+k\frac{d}{dz},$$

where i, j, k are unit vectors parallel to x, y, z respectively. The result of performing this operation twice on any subject is the well known operation (of Laplace)

$$\nabla^2=-\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right).$$

The discovery of the square root of this operation is due to Hamilton; but most of the applications here given, and the whole development of the theory of this operation, are due to Prof. Tait, and are given in several papers, of which the first is in the Proceedings of the Royal Society of Edinburgh, April 28, 1862, and the most complete is that on "Green's and other allied Theorems," Trans. R. S. Edinb. 1869–70.

And, first, I propose to call the result of ∇^2 (Laplace's operation with the negative sign) the *Concentration* of the quantity to which it is applied.

For if Q be a quantity, either scalar or vector, which is a function of the position of a point; and if we find the integral of Q taken throughout the volume of a sphere whose radius is r; then, if we divide this by the volume of the sphere, we shall obtain $\overline{Q}$, the *mean* value of Q within the sphere. If Q_0 is the value of Q at the centre of the sphere, then, when r is small,

$$Q_0-\overline{Q}=\frac{r^2}{10}\nabla^2 Q,$$

or the value of Q at the centre of the sphere exceeds the mean value of Q within the sphere by a quantity depending on the radius, and on $\nabla^2 Q$. Since, therefore, $\nabla^2 Q$ indicates the excess of the value of Q at the centre above its mean value in the sphere, I shall call it the concentration of Q.

If Q is a scalar quantity, its concentration is, of course, also scalar. Thus, if Q is an electric potential, $\nabla^2 Q$ is the density of the matter which produces the potential.

If Q is a vector quantity, then both Q_0 and $\overline{Q}$ are vectors, and $\nabla^2 Q$ is also a vector, indicating the excess of the uniform force Q_0 applied to the whole substance of the sphere above the resultant of the actual force Q acting on all the parts of the sphere.

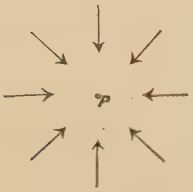
Let us next consider the Hamiltonian operator ∇. First apply it to a scalar function P. The quantity ∇P is a vector, indicating the direction in which P decreases most rapidly, and measuring the rate of that decrease. I venture, with much diffidence, to call this the *slope* of P. Lamé calls the *magnitude* of ∇P the "first differential parameter" of P, but its *direction* does not enter into his conception. We require a vector word, which shall indicate both direction and magnitude, and one not already employed in another mathematical sense. I have taken the liberty of extending the ordinary sense of the word slope from topography, where only two independent variables are used, to space of three dimensions.

If σ represents a vector function, $\nabla\sigma$ may contain both a scalar and a vector part, which may be written $S\nabla\sigma$ and $V\nabla\sigma$.

I propose to call the scalar part the *Convergence* of σ, because, if a closed surface be described about any point, the surface integral of σ, which expresses the effect of the vector σ considered as an inward flux through the surface, is equal to the volume integral of $S\nabla\sigma$ throughout the enclosed space. I think, therefore, that the convergence of a vector function is a very good name for the effect of that vector function in carrying its subject inwards towards a point.

But $\nabla\sigma$ has, in general, also a vector portion, and I propose, but with great diffidence, to call this vector the *Curl* or *Version* of the original vector function.

It represents the direction and magnitude of the rotation of the subject matter carried by the vector σ. I have sought for a word which shall neither, like Rotation, Whirl, or Twirl, connote motion, nor, like Twist, indicate a helical or screw structure which is not of the nature of a vector at all.



CONVERGENCE.

CURL.

CONVERGENCE AND CURL

If we subtract from the general value of the vector function σ its value σ_0 at the point P, then the remaining vector $\sigma - \sigma_0$ will, when there is pure convergence, point towards P. When there is pure curl, it will point tangentially round P; and when there is both convergence and curl, it will point in a spiral manner.

The following statements are true:—

The slope of a scalar function has no curl.

The curl of a vector function has no convergence.

The convergence of the slope of a scalar function is its concentration.

The concentration of a vector function is the slope of its convergence, together with the curl of its curl.

The quaternion expressions, of which the above statements are a translation, were given by Prof. Tait, in his paper in the Proceedings of the Royal Society of Edinburgh, April 28th, 1862; but for the more complete mathematical treatment of the operator ∇, see a very able paper by Prof. Tait, "On Green's and other Allied Theorems," (Transactions of the Royal Society of Edinburgh, 1870,) and another paper in the Proceedings R. S. Edinb. for 1870–71, p. 318.

Mr. Tucker also read an Historical Note, by Mr. Todhunter, F.R.S., upon certain formulæ in Spherical Trigonometry, in which the writer claimed for Delambre the authorship of the formulæ generally known as Gauss's Analogies.

The following presents were received:—

"Crelle's Journal," 73 Band, 1es Heft, Jan. 1871.

"Annali di Matematica," Tom iv., Fasc. i., July 1870.

And the following Memoirs by M. Michel Chasles, presented by Dr. Hirst:

"Mémoire sur l'attraction des ellipsoïdes: solution synthétique pour le cas général d'un ellipsoïde hétérogène et d'un point extérieur." (1846.)

"Théorèmes généraux sur l'attraction des corps." (1845.)

"Rapport sur un Mémoire de M. Mannheim intitulé: sur le déplacement d'une figure de forme invariable. Nouvelle méthode des Normales: applications diverses. Rapporteur M. Chasles." (1868.)

"Sur les systèmes de courbes d'ordre quelconque. Courbes exceptionnelles." 2 copies. (1867.)

"Recherche des traces anciennes du système de l'Abacus. Calcul de Victorius et Commentaire d'Abbon." (1867.)

"Remarques sur les questions de contact de courbes d'ordre quelconque avec une courbe donnée dont les points se déterminent individuellement." (1866.)

"Relations entre les deux caractéristiques d'un système de courbes d'ordre quelconque." (1866.)

"Systèmes de coniques qui satisfont à sept conditions dans l'espace." 2 copies. (1865.)

"Nouvelle solution du problème de l'attraction d'un ellipsoïde hétérogène sur un point extérieur." (1840.)

"Construction des racines des équations du troisième et du quatrième ordre." (1855.)

April 13th, 1871.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Mr. C. J. Monro, B.A., late Fellow of Trinity College, Cambridge, was proposed for election.

The Hon. J. W. Strutt, M.A., and Mr. James Stuart, B.A., Fellows of Trinity College, Cambridge, and Major Frederick Close, R.A., were elected members.

Before proceeding to the reading of papers, the President briefly alluded to the great loss the Society had sustained in consequence of the death of its first President, the late Prof. De Morgan. Dr. Hirst, who had been well acquainted with Mr. De Morgan for some years, having been associated with him as a Professor in University College, gave a sketch of the work done by the deceased, especially dwelling upon the originality and acuteness displayed in all his writings, pointing out the fact, that in his Double Algebra was perhaps to be found the first trace of what was subsequently expanded by Sir W. R. Hamilton into the Theory of Quaternions; in the treatise on the Differential and Integral Calculus were to be found the germs of much of what has subsequently come prominently forth under the hands of Boole and others; and dwelling on the ability displayed in the Formal Logic. The speaker then mentioned how great an interest Mr. De Morgan had taken in the Society since its very commencement, which led him to undertake the office of President, and to deliver the inaugural address.

Dr. Hirst then, after stating that it was Mr. De Morgan who further did away with the original restriction of membership to persons associated with University College, paid a cordial and feeling tribute to the personal character of the deceased.

Prof. Crofton, F.R.S., then explained his Diagrams illustrative of the "Stresses in Warren and Lattice Girders." He stated that he had not found anything to help him in English text-books, and referred to papers by Profs. Rankine and Clerk-Maxwell. Prof. Henrici and Mr. Merrifield, in their remarks on the communication, drew attention to a

work little known in this country, by Culmann (Die Graphische Statik, 1866), in which Mr. Crofton's constructions had been anticipated, and the methods applied to a very wide range of subjects. Prof. Henrici further illustrated the subject of Prof. Crofton's remarks by a very simple and ingenious notation.

Prof. Cayley followed with a brief sketch of the contents of the following paper:—

A Third Memoir on Quartic Surfaces.

The present Memoir is a continuation of my former researches on Nodal Quartic Surfaces. The leading idea is, that for a quartic surface with k-nodes, given the nature of the circumscribed, $(k-1)$ nodal, sextic cone belonging to any one node of the surface [for instance, $k=10$, that it is a cone (3, 3) composed of two cubic cones], we thereby determine the equation of the quartic surface, and consequently the nature of the remaining $(k-1)$ nodes thereof. By means of this general theory I complete, in an essential point, the theory of the *Symmetroid;* viz., I show that a 10-nodal quartic surface having a single node (3, 3) is a Symmetroid; whence, as appears by my second Memoir, each of the remaining nine nodes is also a node (3, 3); and we have the theory of the remarkable system of ten points in space such that, joining any one of them with the remaining nine, the nine lines thus obtained are the intersections of two cubic cones. A large part of the Memoir is devoted to the consideration of the surfaces with 16, 15, 14, and 13 nodes: this is substantially a reproduction of the results obtained by Kummer in the Memoir "Ueber die algebraischen Strahlensysteme, &c.," already referred to; but the results in question are brought into connexion with the theory of the present Memoir, and they are, by a change of the constants, exhibited in a form of much greater symmetry and elegance. I attach importance also to the square diagrams by means of which I have exhibited, in a compendious form, the relation between the several nodes and circumscribed sextic cones.

The paragraphs are numbered consecutively with those of the first and second Memoirs.

Preliminary Considerations and Classification.

125. I call to mind that if a quartic surface has a node (conical point), then there is for this node a tangent quadricone and a circumscribed sextic cone; viz., if the surface has $(k-1)$ other nodes, or in all k nodes, then the sextic cone has $(k-1)$ nodal lines (passing through the other nodes respectively), and we have thus for the different forms of the sextic the table p. 25; viz., this is

Nodes of Surface.	Nodal Lines of Cone.	Circumscribed Sextic Cone.							
1	0	6							
2	1	6_1							
3	2	6_2							
4	3	6_3							
5	4	6_4							
6	5	6_5	$5, 1$						
7	6	6_6	$5_1, 1$						
8	7	6_7	$5_2, 1$						
9	8	6_8	$5_3, 1$	$4, 2$					
10	9	6_9	$5_4, 1$	$4_1, 2$	$4, 1, 1$	$3, 3$			
11	10	6_{10}	$5_5, 1$	$4_2, 2$	$4_1, 1, 1$	$3_1, 3$			
12	11	...	$5_6, 1$	$4_3, 2$	$4_2, 1, 1$	$3_1, 3_1$	$3, 2, 1$		
13	12	...	...	...	$4_3, 1, 1$	...	$3_1, 2, 1$	$3, 1, 1, 1$	$2, 2, 2$
14	13	...	...	...	...	...	...	$3_1, 1, 1, 1$	$2, 2, 1, 1$
15	14	...	...	...	...	...	...	...	$2, 1, 1, 1, 1$
16	15	...	...	...	...	...	...	...	$1, 1, 1, 1, 1, 1$

viz., 6 denotes a proper sextic cone without nodal lines; 6_1 a proper sextic cone with one nodal line; 5, 1 a proper quintic cone and a plane, &c.

We may distinguish the nodes according to the sextic cones; thus, a node 6 means a node for which the circumscribed cone is a proper sextic cone, $(1, 1, 1, 1, 1, 1)$ a node where the circumscribed cone breaks up into six planes, &c.

126. A 16-nodal surface has 16 nodes $(1, 1, 1, 1, 1, 1)$, and a 15-nodal surface has 15 nodes $(2, 1, 1, 1, 1)$; but, for a 14-nodal surface, the question arises how many nodes are $(3_1, 1, 1, 1)$, and how many $(2, 2, 1, 1)$. It was remarked, p. 26, that the only possible cases were 14, 0; 8, 6; or 2, 12; and that we might, in like manner, limit the number of possible cases for other values of k; but that the inquiry was not then further pursued. I resume this inquiry, but without obtaining as yet a complete answer.

127. It is to be observed that a line joining any two nodes is not, in general, a line on the surface, but that it may be so; the surfaces for which this is so (viz., any surface which contains upon it a line through two nodes) form, however, a class by themselves, which at present I *altogether exclude from consideration.* This being so, it will appear in the sequel that there is but one kind of surface having a node $(2, 2, 1, 1)$, and but one kind of surface having a node $(3_1, 1, 1, 1)$. Now there is a surface, Kummer's 14-nodal, the nodes of which are $8\,(3_1, 1, 1, 1) + 6\,(2, 2, 1, 1)$; wherefore the two kinds are identical, and

are each of them Kummer's 14-nodal surface. Similarly, for the 13-nodal surfaces, there is but one kind having a node $(4_3, 1, 1)$, but one kind having a node $(3, 1, 1, 1)$, and but one kind having a node $(3_1, 2, 1)$; and we have Kummer's 13-nodal surface with the nodes $3(4_3, 1, 1) + 1(3, 1, 1, 1) + 9(3_1, 2, 1)$; hence the three kinds are identical with each other and with Kummer's. Moreover, there is but one kind having a node $(2, 2, 2)$; hence all the other nodes must be $(2, 2, 2)$, and we have a surface $13(2, 2, 2)$ not given by Kummer. And in like manner for the 12-nodal surfaces, we have the two kinds given by Kummer, and a third kind $12(4_2, 1, 1)$ not given by him; the arrangement thus far being

No. of Nodes.	Character of Surface.
16	$16(1, 1, 1, 1, 1, 1)$,
15	$15(2, 1, 1, 1, 1)$,
14	$8(3_1, 1, 1, 1) + 6(2, 2, 1, 1)$,
13 (α)	$3(4_3, 1, 1) + 1(3, 1, 1, 1) + 9(3_1, 2, 1)$,
„ (β)	$13(2, 2, 2)$,
12 (α)	$12(4_3, 2)$,
„ (β)	$2(5_5, 1) + 6(3_1, 3_1) + 4(3, 2, 1)$,
„ (γ)	$12(4_2, 1, 1)$.

128. But in the next following case we have Kummer's surface, viz.

11 (α) $\quad 1(6_{10}) + 10(3_1, 3)$,

and I do not know whether one, two, or three kinds of surface having nodes $(4_1, 1, 1)$, $(4_2, 2)$, and $(5_5, 1)$. And in the next case we have (as will appear) the Symmetroid, viz.

10 (α) $\quad 10(3, 3)$,

and I do not know how many kinds of surfaces having a node or nodes 6_9, $(5_4, 1)$, $(4_1, 2)$, $(4, 1, 1)$.

It will be observed that the present division has nothing to do with the octadic and dianomal division in the former Memoir.

129. I consider a conic $A=0$, and any six tangents thereof, $t_1=0$, $t_2=0$, $t_3=0$, $t_4=0$, $t_5=0$, $t_6=0$; we have an identical equation which might be written $AC-B^2 = t_1t_2t_3t_4t_5t_6$, but it will be convenient, introducing a constant factor K, to write it

$$AC-B^2 = K t_1t_2t_3t_4t_5t_6,$$

B being a cubic function and C a quartic function of the coordinates.

Consider now the series of factors, such as

$$\begin{aligned} &t_1, \\ &lA + m t_1t_2, \\ &sA + m t_1t_2t_3, \\ &UA + m t_1t_2t_3t_4, \\ &VA + m t_1t_2t_3t_4t_5, \\ &WA + m t_1t_2t_3t_4t_5t_6, \end{aligned}$$

where m is a constant, l a constant, s a linear function, U, V, W functions of the degrees 2, 3, 4 respectively; and compose with one or more such factors an expression involving the term $Kt_1t_2t_3t_4t_5t_6$; for instance, such an expression is

$$\frac{K}{mm'}(sA+mt_1t_2t_3)(l'A+m't_4t_5)t_6;$$

this is, of the form $A\Omega + Kt_1t_2t_3t_4t_5t_6$, viz., $A\Omega + (AC-B^2)$, or $A(\Omega+C)-B^2$, say $A\Gamma-B^2$; or what is the same thing, introducing a new coordinate w, we have a quadric function

$$Aw^2+2Bw+\Gamma,$$

the discriminant of which, $A\Gamma-B^2$, is equal to the expression in question.

130. In the sequel (x, y, z, w) are considered as the coordinates of a point in space; $A=0$ is thus a quadric cone, $t_1=0$, $t_2=0 \ldots t_6=0$, any six tangent planes thereof; and hence $Aw^2+2Bw+\Gamma=0$ a quartic surface, having the point $(x=0, y=0, z=0)$ for a node, whereof the circumscribed cone $A\Gamma-B^2=0$ breaks up in the assumed manner.

Thus, in order that the circumscribed cone may be as above

$$(sA+mt_1t_2t_3)(l'A+m't_4t_5)t_6,$$

we have only to assume

$$\Gamma = C + \frac{K}{mm'}(sl'A+sm't_4t_5+l'mt_1t_2t_3)t_6,$$

and so in other cases. Observe that $sA+mt_1t_2t_3=0$ is a cubic cone, which, so long as s, m are arbitrary, has no nodal line; but establishing a single relation (say s remains arbitrary, but a proper value is assigned to m) it will be a cubic cone having a nodal line. And so $UA+mt_1t_2t_3t_4=0$ is a quartic cone without any nodal line, but by particularising the constants it may be made to have one, two, or three nodal lines. Such nodal determinations are obviously required in order that the formula may extend to all the before-mentioned forms of the circumscribed cone. The foregoing analysis is the foundation of the whole theory: I have given it, as above, apart from the theory, in order that the nature of it may be the better perceived; but I have now to bring it into connexion with the theory.

On the Sextic Curves $A_2B_4-C_3^2=0$.

131. I revert to the consideration of plane curves. The equation of a sextic curve $(*\,\mathfrak{X}x, y, z)^6=0$ cannot be in general expressed in the form $AC-B^2=0$, where the degrees of A, B, C are 2, 3, 4 respectively; in fact, the existence of such a form implies that there is a conic $A=0$ touching the sextic 6 times; and since a conic can only be made to satisfy 5 conditions, there is not in general any such conic.

132. Such conic, when it exists, is said to be inscribed in the sextic,

and the sextic to be circumscribed about the conic, or to be an "amphigram"; and then, $A=0$ being the equation of the conic, that of the sextic is expressible in the form in question $AC-B^2=0$. It is clear that $B=0$ is a cubic curve passing through the 6 points of contact of the conic with the sextic, and that any such curve may be taken for the curve B; in fact if a particular cubic through the 6 points is $B'=0$, and the equation of the sextic is $AC'-B'^2=0$, then taking p an arbitrary linear function of the coordinates, the equation of the general cubic is $B=B'+pA=0$; and then writing

$$A=A,$$
$$B=B'+pA,$$
$$C=C'+2B'p+Ap^2,$$

we have $AC-B^2=AC'-B'^2$; so that the original form $AC'-B'^2=0$ becomes $AC-B^2=0$. But the cubic $B=0$ being assumed at pleasure, the quartic $C=0$ is a determinate curve.

133. It is to be observed that a sextic curve may be an amphigram in more than one way: certainly in two, three, or four, and possibly in a greater number of ways. For the equation of the curve contains 27 constants, and hence determining the sextic so as to touch 4 given conics each of them 6 times, there are still 3 constants; and the curve will be an amphigram in regard to each of the 4 conics; say it is a quadruple amphigram. But in the sequel we are only concerned with a sextic curve considered as an amphigram in regard to a given conic $A=0$ (no attention being paid to the other inscribed conics, if any); and then, by what precedes, taking $B=0$ any cubic whatever through the 6 points of contact, we have a determinate quartic curve $C=0$, and the equation of the sextic curve assumes the form $AC-B^2=0$.

134. The curves $A=0$, $B=0$, $C=0$ contain respectively 5, 9, 14 constants; whence considering the function B as containing an arbitrary constant factor, for the curve $AC-B^2=0$, the number of constants is *primâ facie* $5+9+14+1=29$; but on account of the arbitrary linear function p, the real number is $29-3=26$: this is right, for a sextic curve contains 27 constants; and the curve being an amphigram, there is one relation between the constants, $27-1=26$.

135. Suppose now that the sextic curve $AC-B^2=0$ breaks up into two or more separate curves, say into the two curves $P=0$, $Q=0$ of the orders f, g respectively; $f+g=6$. We have

$$AC-B^2=PQ=0;$$

and the conic $A=0$ touching the sextic six times, must, it is clear, touch the curves $P=0$, $Q=0$, f and g times respectively. And so when the sextic breaks up into any number of curves, each component curve $P=0$ of the order f must touch the sextic g times.

136. It follows that if the sextic break up into six lines, say $AC-B^2 = t_1t_2t_3t_4t_5t_6 = 0$, then that each of the lines $t_1=0$, $t_2=0$, ... $t_6=0$ is a tangent to the conic. And conversely, starting with the conic $A=0$ and any six tangents thereof $t_1=0$, $t_2=0$, ... $t_6=0$, we have an identity of the form in question. In fact, taking any two of the tangents, say $t_1=0$ and $t_2=0$, then, if $p=0$ be the equation of the line joining their points of intersection, the equation of the conic will be of the form $t_1t_2+p^2=0$, that is, we may write $A=t_1t_2+p^2$, or what is the same thing, $t_1t_2 = A-p^2$. (Considering A as a given quadric function of the coordinates, this of course implies that the implicit constant factors of t_1, t_2, p are properly determined.) Similarly, $q=0$ being the line through the points of contact of t_3, t_4, and $r=0$ that through the points of contact of t_5, t_6, we have $t_3t_4 = A-q^2$ and $t_5t_6 = A-r^2$; whence, to satisfy the equation

$$AC-B^2 = t_1t_2t_3t_4t_5t_6,$$

we have only to assume $B = lA+pqr$, l an arbitrary linear function of the coordinates, and the equation then gives

$$C = A^2-A(p^2+q^2+r^2)+(q^2r^2+r^2p^2+p^2q^2)+l^2A+2lpqr.$$

137. It will be observed that the grouping of the six tangents into pairs is arbitrary. By altering this grouping, we merely alter the linear function l, but do not obtain any new solution. Thus, say that the new form is $B = l'A+p'q'r'$, then, by properly determining the linear function l, we can reduce this to the original form $B = lA+pqr$; viz., we can satisfy identically the equation $(l-l')A+pqr-p'q'r' = 0$; or what is the same thing, $\lambda A+pqr-p'q'r' = 0$, where λ is a linear function of the coordinates. We have, in fact, the conic $A=0$ and the cubic $pqr=0$ intersecting in the six points of contact any other cubic through these six points; and consequently the cubic $p'q'r' = 0$ must be expressible in the form $\lambda A+pqr=0$, and we have thus the identity in question.

138. We have just seen that the value of B is necessarily of the form $B = lA+pqr$, but we are not concerned with its expression in this particular form. What we require in the sequel is a value of B, and thence one of C, satisfying the identity in question, $AC-B^2 = t_1t_2t_3t_4t_5t_6$; or what is the same thing, introducing for convenience a constant factor K, the identity

$$AC-B^2 = Kt_1t_2t_3t_4t_5t_6.$$

139. Instead of C, I write Γ, and consider the sextic amphigram $A\Gamma-B^2=0$ touched by the conic $A=0$ in the points of contact of the conic with the six tangents $t_1=0$, $t_2=0$, ... $t_6=0$. Suppose the sextic curve breaks up into factors; if one of these factors is a line, it is one of the six tangents, say the tangent $t_1=0$. If there is a conic factor, this is a conic touching the conic $A=0$ at its points of contact with

two of the tangents, say the equation is $lA+mt_1t_2=0$. Similarly, if there is a cubic, quartic, or quintic factor, then the equation hereof is $sA+mt_1t_2t_3=0$, $UA+mt_1t_2t_3t_4=0$, or $VA+mt_1t_2t_3t_4t_5=0$. Or going on to the next case of a sextic factor (being of course the whole curve), we may say that this is $WA+mt_1t_2t_3t_4t_5t_6=0$. (Observe that since $AC-B^2=Kt_1t_2t_3t_4t_5t_6$, this means only that the equation of the sextic amphigram is of the assumed form $A\Gamma-B^2=0$.)

140. By what precedes we can, for a sextic amphigram which breaks up in any assigned manner, determine the value of Γ. For instance, let the amphigram break up into two cubic curves; say these are $sA+mt_1t_2t_3=0$, $s'A+m't_4t_5t_6=0$. Assume

$$A\Gamma-B^2=\frac{K}{mm'}(sA+mt_1t_2t_3)(s'A+m't_4t_5t_6),$$

then this equation is

$$A\Gamma-B^2=\frac{K}{mm'}\{ss'A^2+(sm't_4t_5t_6+s'mt_1t_2t_3)A\}+AC-B^2;$$

that is, we have $\Gamma=C+\frac{K}{mm'}(ss'A+sm't_4t_5t_6+s'mt_1t_2t_3)$,

and so in any other case.

I have already adverted to the question of the "nodal determination" of the formulæ, and it might be properly here considered; viz., the question is as to the determination of the constants in such manner that, for instance, $sA+mt_1t_2t_3=0$ may be a nodal cubic, $UA+mt_1t_2t_3t_4=0$ a nodal, binodal, or trinodal quartic, &c.; but I defer it for the moment in order first to apply the theory to the quartic surfaces.

Application to Quartic Surfaces.

141. If a quartic surface has a node or nodes, we may take for a node the point $x=0$, $y=0$, $z=0$; the equation of the quartic surface is then of the form $Aw^2+2Bw+\Gamma=0$,

where A, B, Γ are functions of x, y, z of the degrees 2, 3, 4 respectively. $A=0$ is the tangent quadricone at the node in question; and the circumscribed cone is $A\Gamma-B^2=0$. By what precedes, this is an amphigram touching the quadricone along six generating lines thereof; say the tangent planes of the cone $A=0$ along these six lines respectively are $t_1=0$, $t_2=0$, ... $t_6=0$. We have then an identical equation

$$AC-B^2=Kt_1t_2t_3t_4t_5t_6,$$

viz., regarding for a moment this equation as an equation for the determination of B, and B′ as any particular solution thereof, then its general solution is $B'+tA$, where t is an arbitrary linear function of (x, y, z), and the B in the equation of the surface is properly $=B'+tA$. But by the substituting $w-t$ in place of w, the B of the equation of the

surface would then be made $= B'$; and it thus appears that we may, without loss of generality, take the B of this equation to be *any particular value* satisfying the identity in question; and then, B having such particular value, C is a quartic function of (x, y, z) completely determined by the same identity. And we then, by what precedes, at once determine Γ so that the circumscribed cone $A\Gamma - B^2 = 0$ may be a cone breaking up in any assigned manner; for instance, if it be a cone (3, 3), then, as just mentioned, the two cubic cones are $sA + mt_1t_2t_3 = 0$, $s'A + m't_4t_5t_6 = 0$; and Γ has the value

$$\Gamma = C + \frac{K}{mm'}(ss'A + sm't_4t_5t_6 + s'mt_1t_2t_3),$$

above obtained.

On the Nodal Determination.

142. I am not able to discuss with much completeness the question of nodal determination. We have to consider a cubic curve $sA + mt_1t_2t_3 = 0$, a quartic curve $UA + mt_1t_2t_3t_4 = 0$, &c., as the case may be, and to determine the constants so that this shall have a node or nodes. Consider for a moment the form $PA + t_1Q = 0$, where Q denotes the product $mt_2t_3 \ldots$ of all or any of the tangents $t_2, \ldots t_5$; the orders of PA, t_1Q are of course equal, that is, the order of P is less by unity than that of Q. I say that, by establishing a single relation between the constants, this may be made to have a node *at the point of contact* $A=0$, $t_1=0$. In fact, writing $\Delta = \lambda\delta_x + \mu\delta_y + \nu\delta_z$, where λ, μ, ν are arbitrary, there will be a node at any point if for that point $\Delta(PA + t_1Q) = 0$. But for the point $A=0$, $t_1=0$ this becomes $P\Delta A + Q\Delta t_1 = 0$; moreover, if $t=0$ be any other tangent of the conic $A=0$, and if $p=0$ be the line joining the points of contact of the tangents t, t_1, then we may write $A = tt_1 - p^2$, and thence (since at the point in question, $A=0$, $t_1=0$, we have also $p=0$) we find $\Delta A = t\Delta t_1$, and the foregoing equation thus becomes $(tP + Q)\Delta t_1 = 0$; viz., this equation is satisfied irrespectively of the values of λ, μ, ν, if only at the point in question (that is, for the values of the coordinates which belong to the point $t_1=0$, $A=0$) we have $tP + Q = 0$, which is a single relation between the constants.

143. In particular the cubic curve $sA + mt_1t_2t_3 = 0$ may be made to have a node at the point of contact of any one of the three tangents; the quartic curve $UA + mt_1t_2t_3t_4 = 0$, a node, or two or three nodes, at the point or points of contact of any one, two, or three of the four tangents; and so in other cases. These are not the only solutions, and they are in fact solutions which (as afterwards explained) I propose to reject, attending in each case only to the remaining or proper solutions of the problem.

144. To obtain in a different manner the foregoing result, consider

again the cubic curve $s\mathrm{A}+mt_1t_2t_3=0$; regarding this as a given curve, the conic $\mathrm{A}=0$ is a conic determined (not of course completely) as a conic having therewith 3 points of 2-pointic intersection; viz., if the cubic has a node, then the cone $\mathrm{A}=0$ is either a conic passing through the node and besides touching the curve twice, or else it is a conic touching the curve 3 times; the former is of course the above mentioned case where there is a node at one of the points of contact on the conic $\mathrm{A}=0$; the latter is regarded as the proper solution. So in the case of a quartic curve $\mathrm{UA}+mt_1t_2t_3t_4=0$, regarding this as a given curve, the conic $\mathrm{A}=0$ is a conic having therewith four points of 2-pointic intersection; viz., if the quartic curve has one, two, or three nodes, then the conic is either a conic passing through one, two, or three nodes, and besides touching the quartic thrice, twice, or once; or else it is a conic touching the quartic four times. The former is the above mentioned case where there is a node or nodes at a point or points of contact with the conic $\mathrm{A}=0$; the latter is regarded as the proper solution.

145. To fix the ideas, and at the same time obtain a result which will be afterwards useful, I work out the formulæ for the cubic curve $s\mathrm{A}+mt_1t_2t_3=0$, taking this equation under the form

$$\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}\right)(x^2+y^2+z^2-2yz-2zx-2xy)+m\,xyz=0.$$

This may have a node in two different ways; viz.,

1° At the point of contact of one of the tangents $x=0$, $y=0$, $z=0$ with the conic $\mathrm{A}=0$; say at the point of contact of $x=0$, that is, the point $x=0$, $y-z=0$. The value of m is $=\frac{4}{\mu}+\frac{4}{\nu}$; hence $\frac{1}{\nu}=-\frac{1}{\mu}+\frac{1}{4}m$; and, substituting, the equation of the curve becomes

$$\left(\frac{x}{\lambda}+\frac{y-z}{\lambda}\right)\{x^2-2x(y+z)+(y-z)^2\}+\frac{1}{4}mz(x+y-z)^2=0,$$

which has obviously a node at the point in question.

2° The node may be at a point not on the conic $\mathrm{A}=0$, viz. the value of m is $=\frac{(\lambda+\mu+\nu)^2}{\lambda\mu\nu}$, the equation is

$$\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}\right)(x^2+y^2+z^2-2yz-2zx-2xy)+\frac{(\lambda+\mu+\nu)^2}{\lambda\mu\nu}xyz=0.$$

In fact, writing for shortness,

$$\begin{aligned}-\lambda+\mu+\nu&=\mathrm{L},\\ \lambda-\mu+\nu&=\mathrm{M}\\ \lambda+\mu-\nu&=\mathrm{N}\\ \lambda+\mu+\nu&=\mathrm{P},\end{aligned}$$

the node is at the point $x:y:z=\mathrm{L}\lambda:\mathrm{M}\mu:\mathrm{N}\nu$; which is at once verified,

if we remark that, writing for convenience $x, y, z = L\lambda, M\mu, N\nu$, then we have

$$\begin{aligned} -x+y+z &= MN, \\ x-y+z &= NL, \\ x+y-z &= LM, \end{aligned}$$

$$\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu} = P, \quad x^2+y^2+z^2-2yz-2zx-2xy = -LMNP\ (=A).$$

For, of the three equations for the coordinates of a node, the first is

$$\frac{1}{\lambda}A+\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}\right)2(x-y-z)+\frac{(\lambda+\mu+\nu)^2}{\lambda\mu\nu}yz = 0,$$

that is, for the values in question,

$$-\frac{1}{\lambda}LMNP+P\,(-2MN)+\frac{P^2}{\lambda\mu\nu}MN\mu\nu = 0,$$

that is,

$$-LMNP-2\lambda MNP+P^2MN = 0,$$

or, finally, $-L-2\lambda+P = 0$, which is satisfied; and similarly the other two equations are satisfied.

Quartic Surfaces resumed.

146. Passing now from curves to cones, and to the theory of the quartic surface, suppose that there is a component cone having a nodal line, say the cubic cone $sA+mt_1t_2t_3 = 0$: if the remaining factor is $t_4t_5t_6$, then we have

$$A\Gamma-B^2 = \frac{K}{m}(sA+mt_1t_2t_3)\,t_4t_5t_6.$$

Suppose the nodal line is a line of contact with the cone $A=0$, say its equations are $t_1=0$, $p=0$ (p a linear function), then $sA+mt_1t_2t_3$ is a quadric function ($*\,\mathfrak{Q}t_1, p)^2$, (of course with variable coefficients); hence $A\Gamma-B^2$ is a quadric function; and A being a linear function ($*\,\mathfrak{Q}t_1, p)$, it follows that B is a linear function, and thence that Γ is also a linear function; that is, A, B, Γ are each of them a linear function ($*\,\mathfrak{Q}t_1, p)$, or the line in question (viz. the line of contact $A=0$, $t_1=0$) is a line on the quartic surface $Aw^2+2Bw+\Gamma = 0$. As already mentioned, *I exclude from consideration* the surfaces which have upon them a line through two nodes; that is, I exclude from consideration the case in question where any component cone, or say where the sextic cone, has a nodal line which is a line on the tangent cone $A=0$.

147. Now, excluding the case just referred to, *I assume as a postulate* that there is but one way in which the cubic cone $sA+mt_1t_2t_3 = 0$ can be made to have a nodal line, or the quartic cone $UA+mt_1t_2t_3t_4 = 0$ one, two, or three nodal lines &c., as the case may be. It is to be understood that this does not mean that the constants are in any of these cases completely determined, but that there is between them a relation or relations constituting a general solution which includes in itself every particular solution whatever. I have no doubt that as

regards the cubic cone at least the assumption is correct. This being so, the character of a single node determines the nature of the surface; for instance, if there is a node $(3_1, 3)$, then taking this as the point $(x=0,\ y=0,\ z=0)$ the equation of the surface is $Az^2+2Bw+\Gamma=0$,

where $$\Gamma = C + \frac{K}{mm'}(ss'A+l's't_4t_5t_6+ls't_1t_2t_3),$$

a surface of a determinate nature; so that the character of all the remaining nodes is completely determined.

148. The point to be attended to is, that if for instance there were two essentially distinct ways of giving the cubic cone $sA+mt_1t_2t_3=0$ a nodal line (such as there would be if the excluded case were considered admissible), then the foregoing equation of the surface would or might include two distinct forms of equation applying to different kinds of surface. The conclusion is that there is but one kind of quartic surface having a node $(3_1, 3)$. Admitting this, and similarly that there is but one kind of quartic surface having a node 6_{10}, it follows that if (as the fact is) there is a surface having the nodes $1(6_{10})+10(3_1, 3)$ (Kummer's 11-nodal surface), then that the two first-mentioned kinds are in fact each of them this last mentioned kind of surface; and it was in this manner that I arrived at the enumeration given near the beginning of the present Memoir.

149. The reasoning is, of course, in place of a direct demonstration which would consist in showing that a surface having a node $(3_1, 3)$ has 9 other like nodes, and also a node 6_{10}; and that a surface having a node 6_{10} has 10 other nodes $(3_1, 3)$; and that, starting from either form of equation, we could, by passing to a node of the other kind, obtain the other form of equation.

Enumeration of the Cases.

150. I collect the results as follows: I call to mind that we have always the identical equation $AC-B^2=Kt_1t_2t_3t_4t_5t_6$, that the equation of the surface is $Aw^2+2Bw+\Gamma=0$, and that the circumscribed cone is $A\Gamma-B^2=0$. The equation of a surface having different kinds of nodes will assume different forms according as the origin (or point $x=0,\ y=0,\ z=0$) is taken to be at a node of one or other of these kinds; these forms of the equation are distinguished as "node-forms," —viz., we speak of the node-form $(3_1, 3)$ when the origin is a node $(3_1, 3)$, and so in other cases.

The 16-nodal surface	$16\,(1, 1, 1, 1, 1, 1)$,
node-form	$(1, 1, 1, 1, 1, 1)$,
cone is	$t_1t_2t_3t_4t_5t_6=0$,
and	$\Gamma=C$;
viz., equation is	$Aw^2+2Bw+C=0$.

The 15-nodal surface $15\,(2,1,1,1,1)$,

node-form $(2,1,1,1,1)$

cone is $(lA+mt_1t_2)\,t_3t_4t_5t_6=0$,

and $\Gamma = C+\dfrac{Kl}{m}t_3t_4t_5t_4=0.$

The 14-nodal surface $8\,(3_1,1,1,1)+6\,(2,2,1,1)$,

node-form $(3_1,1,1,1)$,

cone is $(sA+mt_1t_2t_3)\,t_4t_5t_6=0$,

where $sA+mt_1t_2t_3=0$ is a nodal cubic 3_1,

and $\Gamma = C+\dfrac{Ks}{m}t_4t_5t_6$;

node-form $(2,2,1,1)$,

cone is $(lA+mt_1t_2)\,(l'A+m't_3t_4)\,t_5t_6=0$,

and $\Gamma = C+\dfrac{K}{mm'}(ll'A+lm't_3t_4+l'mt_1t_2)\,t_5t_6.$

The 13(α)-nodal surface $3\,(4_3,1,1)+1\,(3,1,1,1)+9\,(3_1,2,1)$,

node-form $(4_3,1,1)$,

cone is $(UA+mt_1t_2t_3t_4)\,t_5t_6=0$,

where $UA+mt_1t_2t_3t_4=0$ is a trinodal quartic 4_3,

and $\Gamma = C+\dfrac{K}{m}Ut_5t_6$;

node-form $(3,1,1,1)$,

cone is $(sA+mt_1t_2t_3)\,t_4t_5t_6=0$,

and $\Gamma = C+\dfrac{K}{m}st_4t_5t_6$;

node-form $(3_1,2,1)$,

cone is $(sA+mt_1t_2t_3)\,(l'A+m't_4t_5)\,t_6=0$,

where $sA+mt_1t_2t_3=0$ is a nodal cubic 3_1,

and $\Gamma = C+\dfrac{K}{mm'}(sl'A+m'st_4t_5+ml't_1t_2t_3)\,t_6.$

The 13(β)-nodal surface $13\,(2,2,2)$,

node-form $(2,2,2)$,

cone is $(lA+mt_1t_2)\,(l'A+m't_3t_4)\,(l''A+m''t_5t_6)=0$,

and $\Gamma = C+\dfrac{K}{mm'm''}\{ll'l''A^2+(l'l''mt_1t_2+l''lm't_3t_4+ll'm''t_5t_6)A$
$+lm'm''t_3t_4t_5t_6+l'm''mt_5t_6t_1t_2+l''mm't_1t_2t_3t_4\}.$

The 12(α)-nodal surface $12\,(4_3,2)$,

node-form $(4_3,2)$,

cone is $(\mathrm{U}A+mt_1t_2t_3t_4)(l'A+m't_5t_6)=0$,

where $\mathrm{U}A+mt_1t_2t_3t_4=0$ is a trinodal quartic 4_3,

and $\Gamma=C+\frac{K}{mm'}(l'\mathrm{U}A+m'\mathrm{U}t_5t_6+l'mt_1t_2t_3t_4)$.

The 12(β)-nodal surface $2(5_6,1)+6(3_1,3_1)+4(3,2,1)$,

node-form $(5_6,1)$,

cone is $(\mathrm{V}A+mt_1t_2t_3t_4t_5)t_6=0$,

where $\mathrm{V}A+mt_1t_2t_3t_4t_5=0$ is a 6-nodal quintic 5_6,

and $\Gamma=C+\frac{K}{m}\mathrm{V}t_6$;

node-form $(3_1,3_1)$,

cone is $(sA+mt_1t_2t_3)(s'A+m't_4t_5t_6)=0$,

where $sA+mt_1t_2t_3=0$ and $s'A+m't_4t_5t_6=0$ are each of them a nodal cubic 3_1,

and $\Gamma=C+\frac{K}{mm'}(ss'A+m'st_4t_5t_6+ms't_1t_2t_3)$;

node-form $(3,2,1)$,

cone is $(sA+mt_1t_2t_3)(l'A+m't_4t_5)t_6=0$,

and $\Gamma=C+\frac{K}{mm'}(l'sA+m'st_4t_5+l'mt_1t_2t_3)t_6$.

The 12(γ)-nodal surface,

node-form $(4_2,1,1)$,

cone is $(\mathrm{U}A+mt_1t_2t_3t_4)t_5t_6=0$,

where $\mathrm{U}A+mt_1t_2t_3t_4=0$ is a binodal quartic 4_2,

and $\Gamma=C+\frac{K}{m}\mathrm{U}t_5t_6$.

The 11(α)-nodal surface,

node-form (6_{10}),

cone is $\mathrm{W}A+mt_1t_2t_3t_4t_5t_6=0$,

where this is a 10-nodal sextic 6_{10},

and $\Gamma=C+\frac{K}{m}\mathrm{W}$;

node-form $(3_1,3)$,

cone is $(sA+mt_1t_2t_3)(s'A+m't_4t_5t_6)=0$,

where $sA+mt_1t_2t_3=0$ is a nodal cubic 3_1,

and $\Gamma=C+\frac{K}{mm'}(ss'A+m'st_4t_5t_6+ms't_1t_2t_3)$.

Other 11-nodal surfaces,

node-form $(5_5,1)$,

cone is $(\mathrm{V}A+mt_1t_2t_3t_4t_5)t_6=0$,

where $\nabla A + mt_1t_2t_3t_4t_5 = 0$ is a 5-nodal quintic 5_5,

and $$C = \Gamma + \frac{K}{m}\nabla t_6 = 0;$$

node-form $(4_2, 2)$,

cone is $(UA + mt_1t_2t_3t_4)(l'A + m't_5t_6) = 0$,

where $UA + mt_1t_2t_3t_4 = 0$ is a binodal quartic 4_2,

and $$\Gamma = C + \frac{K}{mm'}(l'UA + m'Ut_5t_6 + l'mt_1t_2t_3t_4);$$

node-form $(4, 1, 1)$,

cone is $(UA + mt_1t_2t_3t_4)\,t_5t_6 = 0$,

where $UA + mt_1t_2t_3t_4$ is a nodal quartic 4_1,

and $$\Gamma = C + \frac{K}{m}Ut_5t_6;$$

but whether these node-forms belong to the same or to different surfaces is not ascertained.

The enumeration is not extended to the 10-nodal surfaces, but I consider one case of these surfaces.

The $10(a)$*-nodal surface* $10(3, 3)$.

151. I assume only that there is a single node $(3, 3)$: taking the cone to be $$(sA + mt_1t_2t_3)(s'A + m't_4t_5t_6) = 0;$$

then for the equation of the surface, in the node-form $(3, 3)$ in question, we have $$\Gamma = C + \frac{K}{mm'}(ss'A + sm't_4t_5t_6 + s'mt_1t_2t_3).$$

But I present this result under a different form, as follows: I write

$$A = p^2 + ft_1t_2 = q^2 + gt_3t_4 = r^2 + ht_5t_6,$$

where f, g, h are constants, and, as before, $p=0$, $q=0$, $r=0$ are the lines joining the points of contact of t_1, t_2; t_3, t_4; and t_5, t_6 respectively: we have

$$sA + mt_1t_2t_3 = sA + mt_3\left(\frac{A - p^2}{f}\right), \text{ and } s'A + m't_4t_5t_6 = s'A + m't_4\left(\frac{A - r^2}{h}\right);$$

or in place of s, s' introducing new linear functions σ, σ', the cubic curves may be taken to be $\sigma A - \frac{m}{f}p^2t_3$, $\sigma'A - \frac{m'}{h}r^2t_4$, so that we have

$$\begin{aligned} A\Gamma - B^2 &= \frac{K}{mm'}\left(\sigma A - \frac{m}{f}p^2t_3\right)\left(\sigma'A - \frac{m'}{h}r^2t_4\right) \\ &= \frac{K}{mm'}\left(\sigma\sigma'A^2 - \sigma A\frac{m'}{h}r^2t_4 - \sigma'A\frac{m}{f}p^2t_3 + \frac{mm'}{fh}p^2r^2\frac{A - q^2}{g}\right) \\ &= \frac{K}{mm'}\left(\sigma\sigma'A^2 - \sigma A\frac{m'}{h}r^2t_4 - \sigma'A\frac{m}{f}p^2t_3 + \frac{mm'}{fgh}p^2r^2A\right) - \frac{K}{fgh}p^2q^2r^2; \end{aligned}$$

whence $\mathrm{B} = \left(\frac{\mathrm{K}}{fgh}\right)^{\frac{1}{3}}(pqr+t\mathrm{A})$, where t is a linear function of the coordinates; and we then have

$$\Gamma = \frac{\mathrm{K}}{mm'}\left(\sigma\sigma'\mathrm{A} - \sigma\frac{m'}{h}r^2t_4 - \sigma'\frac{m}{f}p^2t_3 + \frac{mm'}{fgh}p^2r^2\right) + \frac{\mathrm{K}}{fgh}(t^2\mathrm{A}+2tpqr),$$

where A may be considered as standing for $q^2+gt_3t_4$. The equation $\mathrm{A}w^2+2\mathrm{B}w+\Gamma=0$ of the surface, substituting throughout for A its value, is therefore

$$(q^2+gt_3t_4)\,w^2 + 2\left(\frac{\mathrm{K}}{fgh}\right)^{\frac{1}{3}}\{pqr+t\,(q^2+gt_3t_4)\}\,w$$
$$+\frac{\mathrm{K}}{mm'}\left[\sigma\sigma'(q^2+gt_3t_4) - \sigma\frac{m'}{h}r^2t_4 - \sigma'\frac{m}{f}p^2t_3 + \frac{mm'}{fgh}p^2r^2\right]$$
$$+\frac{\mathrm{K}}{fgh}[t^2(q^2+gt_3t_4)+2tpqr] = 0,$$

where the cone is

$$\left\{\sigma(q^2+gt_3t_4) - \frac{m}{f}p^2t_3\right\}\left\{\sigma'(q^2+gt_3t_4) - \frac{m'}{h}r^2t_4\right\} = 0.$$

152. Writing in the equation of the surface $w\left(\frac{\mathrm{K}}{fgh}\right)^{\frac{1}{3}}$ instead of w, it becomes

$$(q^2+gt_3t_4)\,w^2 + 2\,[pqr+t\,(q^2+gt_3t_4)]\,w$$
$$+\frac{fgh}{mm'}\left[\sigma\sigma'(q^2+gt_3t_4) - \sigma\frac{m'}{h}r^2t_4 - \sigma'\frac{m}{f}p^2t_3 + \frac{mm'}{fgh}p^2r^2\right]$$
$$+t^2\,(q^2+gt_3t_4) + 2tpqr = 0;$$

and then writing $\frac{m}{f}\sigma$ and $\frac{m'}{h}\sigma'$ for σ and σ' respectively, this is

$$(q^2+gt_3t_4)\,w^2 + 2pqrw + 2tw\,(q^2+gt_3t_4)$$
$$+g\left[\sigma\sigma'(q^2+gt_3t_4) - \sigma r^2t_4 - \sigma'p^2t_3 + \frac{1}{g}p^2r^2\right] + t^2(q^2+gt_3t_4) + 2tpqr = 0.$$

We may consider t_3, t_4 as denoting not the functions originally so represented, but these functions each multiplied by a suitable constant, and thereupon write $g=-1$; viz., $t_3=0$, $t_4=0$, will now denote any two tangents to the conic $\mathrm{A}=0$, the implicit factors being so determined that $\mathrm{A} = q^2-t_3t_4$. The equation of the surface is

$$(q^2-t_3t_4)w^2 + 2pqrw + 2tw(q^2-t_3t_4)$$
$$-\sigma\sigma'(q^2-t_3t_4) + \sigma r^2t_4 + \sigma'p^2t_3 + p^2r^2 + t^2(q^2-t_3t_4) + 2pqrt = 0;$$

viz., this is

$$(q^2-t_3t_4)\left[(w+t)^2-\sigma\sigma'\right] + 2pqr(w+t) + \sigma r^2t_4 + \sigma'p^2t_3 + p^2r^2 = 0.$$

the sextic cone being

$$\{\sigma(q^2-t_3t_4)-p^2t_3\}\{\sigma'(q^2-t_3t_4)-r^2t_4\} = 0.$$

153. But the foregoing equation of the surface is

$$\begin{vmatrix} -\sigma', & w+t, & . , & r \\ w+t, & -\sigma, & p , & . \\ . , & p , & t_4 , & -q \\ r , & . , & -q, & t_3 \end{vmatrix} = 0,$$

as is at once seen by developing the determinant; the functions $w+t$, σ, σ', p, q, r, t_3, t_4 are all of them linear; and the determinant is thus a symmetrical quartic determinant the terms whereof are linear functions of the coordinates; viz. the surface is a symmetroid. That is, a surface having a single node (3, 3) is a symmetroid; but I have shown (*Second Memoir*, p. 200) that a symmetroid has each of its ten nodes (3, 3); wherefore the surface having a single node (3, 3) is the 10 (α)-nodal surface, nodes 10 (3, 3).

154. Start from two cubic cones U=0, V=0, having each the same vertex (x=0, y=0, z=0); we may in a variety of ways determine the two cones αU+βV = 0, γU+δV = 0, having a common inscribed quadric cone A=0 (viz., $\alpha : \beta$ being assumed at pleasure, then $\gamma : \delta$ will be determined; not, I believe, uniquely, but I do not know what the multiplicity is). This being so, the quadric cone A=0 is uniquely determined; and then, assuming at pleasure the plane w=0, the 10 (α)-nodal surface $Aw^2+2Bw+\Gamma=0$ is uniquely determined; consequently the remaining nine nodes are determinate points on the nine lines U=0, V=0 respectively. And we have thus a system of ten points in space such that, joining any one of them with the remaining nine, the nine lines so obtained are the intersections of two cubic cones, or say that they are an ennead of lines.

Notation for the Cases afterwards considered.

155. I proceed to further develope the theory of some of the different surfaces. The same node-form of equation will, of course, assume different shapes according to the actual expressions in terms of the coordinates (x, y, z) of the several functions A, &c., which enter into it. I have found it convenient to attribute to A and B certain specific values which are not in every case those of the coefficients of w^2, w in the equation of the surface: this means that we must, in the equation of the surface, substitute new symbols for these coefficients, and write the equation say in the form $A'w^2+2B'w+\Gamma=0$; the change of notation, when it occurs, will be duly explained.

156. It is in general (but not always) convenient to take the equation of the tangent cone to be $x^2+y^2+z^2-2yz-2zx-2xy=0$; for then any plane $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=0$, where $\alpha+\beta+\gamma=0$, will be a tangent plane; so that six tangent planes may be represented by $x=0$, $y=0$, $z=0$, and by three equations of the form just referred to. And in reference to this assumed form of the equation of the tangent cone, and to what follows, I write

$$\begin{aligned} \alpha\ +\beta\ +\gamma\ &=0, \\ \alpha'+\beta'+\gamma'&=0, \\ \alpha''+\beta''+\gamma''&=0, \end{aligned} \qquad \begin{aligned} \mathrm{X}&=\alpha(\gamma'\gamma''y-\beta'\beta''z), \\ \mathrm{Y}&=\beta(\alpha'\alpha''z-\gamma'\gamma''x), \\ \mathrm{Z}&=\gamma(\beta'\beta''x-\alpha'\alpha''y), \end{aligned}$$

$$\mathrm{P}=\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}, \qquad \begin{aligned} \mathrm{X}'&=\alpha'(\gamma''\gamma y-\beta''\beta z), \\ \mathrm{Y}'&=\beta'(\alpha''\alpha z-\gamma''\gamma x), \\ \mathrm{Z}'&=\gamma'(\beta''\beta x-\alpha''\alpha y), \end{aligned}$$

$$\mathrm{P}'=\frac{x}{\alpha'}+\frac{y}{\beta'}+\frac{z}{\gamma'}, \qquad \begin{aligned} \mathrm{X}''&=\alpha''(\gamma\gamma'y-\beta\beta'z), \\ \mathrm{Y}''&=\beta''(\alpha\alpha'z-\gamma\gamma'x), \\ \mathrm{Z}''&=\gamma''(\beta\beta'x-\alpha\alpha'y), \end{aligned}$$

$$\mathrm{P}''=\frac{x}{\alpha''}+\frac{y}{\beta''}+\frac{z}{\gamma''},$$

$$\mathrm{A}=x^2+y^2+z^2-2yz-2zx-2xy,$$

$$\mathrm{B}=\alpha\alpha'\alpha''(y^2z-yz^2)+\beta\beta'\beta''(z^2x-zx^2)+\gamma\gamma'\gamma''(x^2y-xy^2)+\mathrm{M}xyz,$$

$$\mathrm{C}=(\alpha\alpha'\alpha''yz+\beta\beta'\beta''zx+\gamma\gamma'\gamma''xy)^2;$$

where

$$\begin{aligned} \mathrm{M}&=(\beta-\gamma)\alpha'\alpha''+(\gamma-\alpha)\beta'\beta''+(\alpha-\beta)\gamma'\gamma'' \\ &=(\beta'-\gamma')\alpha''\alpha+(\gamma'-\alpha')\beta''\beta+(\alpha'-\beta')\gamma''\gamma \\ &=(\beta''-\gamma'')\alpha\alpha'+(\gamma''-\alpha'')\beta\beta'+(\alpha''-\beta'')\gamma\gamma' \\ &=-\tfrac{1}{3}\{(\beta-\gamma)(\beta'-\gamma')(\beta''-\gamma'')+(\gamma-\alpha)(\gamma'-\alpha')(\gamma''-\alpha'') \\ &\qquad +(\alpha-\beta)(\alpha'-\beta')(\alpha''-\beta'')\}. \end{aligned}$$

Also $$\mathrm{K}=4\alpha\alpha'\alpha''\beta\beta'\beta''\gamma\gamma'\gamma''.$$

And we have identically

$$\mathrm{AC}-\mathrm{B}^2=\mathrm{K}xyz\left(\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}\right)\left(\frac{x}{\alpha'}+\frac{y}{\beta'}+\frac{z}{\gamma'}\right)\left(\frac{x}{\alpha''}+\frac{y}{\beta''}+\frac{z}{\gamma''}\right).$$

The 16*-nodal surface* 16 (1, 1, 1, 1, 1, 1).

157. Kummer starts from an irrational equation, which is readily converted into the following

$$\sqrt{x(\mathrm{X}-w)}+\sqrt{y(\mathrm{Y}-w)}+\sqrt{z(\mathrm{Z}-w)}=0,$$

and then, rationalizing, we have

$$\mathrm{A}w^2+2\mathrm{B}w+\mathrm{C}=0,$$

where as above $AC-B^2 = K\alpha\beta\gamma PP'P''$.

This agrees with the foregoing theory; viz., the point ($x=0$, $y=0$, $z=0$) being a node, the rationalized equation must, of course, be in the node-form (1, 1, 1, 1, 1, 1), (being the only node-form); and the symmetry of the formulæ enables us at once to write down the equations of the 16 singular planes, and thence to deduce the coordinates of the 16 nodes; viz.,

the singular planes are	and the nodes are
(1) $x=0$	(1) $(0, -\beta, \gamma, \alpha'\alpha''\beta\gamma)$
(2) $y=0$	(2) $(\alpha, 0, -\gamma, \beta'\beta''\gamma\alpha)$
(3) $z=0$	(3) $(-\alpha, \beta, 0, \gamma'\gamma''\alpha\beta)$
(4) $w=0$	(4) $(\alpha'\alpha'', \beta'\beta'', \gamma'\gamma'', 0)$
(5) $X-w=0$	(5) $(1, 0, 0, 0)$
(6) $Y-w=0$	(6) $(0, 1, 0, 0)$
(7) $Z-w=0$	(7) $(0, 0, 1, 0)$
(8) $P=0$	(8) $(0, 0, 0, 1)$
(9) $X'-w=0$	(9) $(0, -\beta', \gamma', \alpha''\alpha\beta'\gamma')$
(10) $Y'-w=0$	(10) $(\alpha', 0, -\gamma', \beta''\beta\gamma'\alpha')$
(11) $Z'-w=0$	(11) $(-\alpha', \beta', 0, \gamma''\gamma\alpha'\beta')$
(12) $P'=0$	(12) $(\alpha''\alpha, \beta''\beta, \gamma''\gamma, 0)$
(13) $X''-w=0$	(13) $(0, -\beta'', \gamma'', \alpha\alpha'\beta''\gamma'')$
(14) $Y''-w=0$	(14) $(\alpha'', 0, -\gamma'', \beta\beta'\gamma''\alpha'')$
(15) $Z''-w=0$	(15) $(-\alpha'', \beta'', 0, \gamma\gamma'\alpha''\beta'')$
(16) $P''=0$	(16) $(\alpha\alpha', \beta\beta', \gamma\gamma', 0)$

where the nodes and planes are numbered as by Kummer; and by means of his (differently arranged) diagram of the relation between the several nodes and planes, I was enabled to form the following square diagram, which exhibits this relation in, I think, the most convenient form. To explain this, observe that in the upper and left-hand margins, the numbers refer to the nodes; in the body of the table, and in the right-hand margin to the planes, the table shows that for the node 1, the circumscribed cone is made up of the planes 1, 6, 7, 8, 9, 13; and that the remaining 15 nodes are situate on the nodal lines of this cone, the node 2 on the intersection of the planes 7, 8; the node 3 on the intersection of the planes 6, 8, and so on; and the like as regards the other lines of the table.

NODES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	CONES
1	*	7,8	6,8	6,7	9,13	1,6	1,7	1,8	1,9	6,13	7,13	8,13	1,13	6,9	7,9	8,9	1, 9, 13, 8, 7, 6
2	7,8	*	5,8	5,7	2,5	10,14	2,7	2,8	5,14	2,10	7,14	8,14	5,10	2,14	7,10	8,10	2, 10, 14, 7, 8, 5
3	6,8	5,8	*	5,6	3,5	3,6	11,15	3,8	5,15	6,15	3,11	8,15	5,11	6,11	3,15	8,11	3, 11, 15, 6, 5, 8
4	6,7	5,7	5,6	*	4,5	4,6	4,7	12,16	5,16	6,16	7,16	4,12	5,12	6,12	7,12	4,16	4, 12, 16, 5, 6, 7
5	9,13	2,5	3,5	4,5	*	3,4	2,4	2,3	5,9	2,13	3,13	4,13	5,13	2,9	3,9	4,9	5, 13, 9, 4, 3, 2
6	1,6	10,14	3,6	4,6	3,4	*	1,4	1,3	1,14	6,10	3,14	4,14	1,10	6,14	3,10	4,10	6, 14, 10, 3, 4, 1
7	1,7	2,7	11,15	4,7	2,4	1,4	*	1,2	1,15	2,15	7,11	4,15	1,11	2,11	7,15	4,11	7, 15, 11, 2, 1, 4
8	1,8	2,8	3,8	12,16	2,3	1,3	1,2	*	1,16	2,16	3,16	8,12	1,12	2,12	3,12	8,16	8, 16, 12, 1, 2, 3
9	1,9	5,14	5,15	5,16	5,9	1,14	1,15	1,16	*	15,16	14,16	14,15	1,5	9,14	9,15	9,16	9, 1, 5, 16, 15, 4
10	6,13	2,10	6,15	6,16	2,13	6,10	2,15	2,16	15,16	*	13,16	15,13	10,13	2,6	10,15	10,16	10, 2, 6, 15, 16, 13
11	7,13	7,14	3,11	7,16	3,13	3,14	7,11	3,16	14,16	13,16	*	13,14	11,13	11,14	3,7	11,16	11, 3, 7, 14, 13, 16
12	8,13	8,14	8,15	4,12	4,13	4,14	4,15	8,12	14,15	13,15	13,14	*	12,13	12,14	12,15	4,8	12, 4, 8, 13, 14, 15
13	1,13	5,10	5,11	5,12	5,13	1,10	1,11	1,12	1,5	10,13	11,13	12,13	*	11,12	10,12	10,11	13, 5, 1, 12, 11, 10
14	6,9	2,14	6,11	6,12	2,9	6,14	2,11	2,12	9,14	2,6	11,14	12,14	11,12	*	9,12	9,11	14, 6, 2, 11, 12, 9
15	7,9	7,10	3,15	7,12	3,9	3,10	7,15	3,12	9,15	10,15	3,7	12,15	10,12	9,12	*	9,10	15, 7, 3, 10, 9, 12
16	8,9	8,10	8,11	4,16	4,9	4,10	4,11	8,16	9,16	10,16	11,16	4,8	10,11	9,11	9,10	*	16, 8, 4, 9, 10, 11

158. The before mentioned irrational equation may be written

$$\sqrt{1.5} + \sqrt{2.6} + \sqrt{3.7} = 0,$$

and by symmetry we see that also

$$\sqrt{1.9} + \sqrt{2.10} + \sqrt{3.11} = 0,$$
$$\sqrt{1.13} + \sqrt{2.14} + \sqrt{3.15} = 0;$$

viz., these are three equations each containing the planes 1, 2, 3, which are three of the planes belonging to the node 1; the other three planes in any such equation (for instance, the planes 5, 6, 7, in the first equation) being three planes belonging to another node. Instead of the planes 1, 2, 3, we may have any other three planes belonging to the node 1; and instead of the node 1, any other node; but each equation belongs to two nodes: the number of equations is thus

$$\frac{6.5.4}{1.2.3} \times 16 \times 3 \div 2, = 480.$$

159. To obtain the planes belonging to any such equation, combine any two of the outside right-hand lines of the diagram, these contain in common two numbers the places of which are interchanged; striking these out, we have four columns, and taking out of these any three columns, we have the corresponding sets of planes. For instance, lines 1 and 2 contain 78 and 87 respectively; striking these out, the lines are

$$1,\ 9,\ 13,\ 6;$$
$$2,\ 10,\ 14,\ 5;$$

whence we have the sets (1, 9, 13) and (2, 10, 14); viz., there is an irrational equation of the form

$$\sqrt{1.2} + \sqrt{9.10} + \sqrt{13.14} = 0,$$

but it is probably necessary to introduce constant factors along with the products 1 . 2, 9 . 10, and 13 . 14 respectively. There are $\frac{1}{2}16.15$, $=120$ pairs of lines, and each line gives 4 equations; in all 120×4, $=480$ equations, as above.

160. I stop to remark that Kummer gives for his 13-nodal surface an equation containing three arbitrary constants, say λ, μ, ν, such that, putting one of these $=0$, we have the 14-nodal surface; putting two of them each $=0$, the 15-nodal surface; putting all three of them $=0$, the 16-nodal surface. The equations for the 16-nodal surface and that for the 14-nodal surface, made use of by Kummer, are, in fact, those deduced as above from the equation of the 13(a)-nodal surface; and the like form might have been used for the 15-nodal surface. But the form actually used by Kummer, as presently appearing, is an equivalent form not thus deducible from the equation of the 13(a)-nodal surface.

The 15*-nodal Surface* $15(2,1,1,1,1)$.

161. Kummer's equation is readily converted into the following:

$$Aw^2+2Bw+C+\frac{Kl}{m}xyzP=0,$$

the circumscribed cone being thus

$$(lA+mP'P'')\,xyzP=0,$$

and the equation being in the node-form $(2,1,1,1,1)$.

The formulæ for the 15 nodes and the 10 singular planes depend upon a quadric equation, for the symmetrical expression of which I write

$$\alpha'\alpha''-\beta''\gamma'=\beta'\beta''-\gamma''\alpha'=\gamma'\gamma''-\alpha''\beta'=\omega,$$
$$\alpha'\alpha''-\beta'\gamma''=\beta'\beta''-\gamma'\alpha''=\gamma'\gamma''-\alpha'\beta''=\varpi;$$

so that

$$\omega-\varpi=\beta'\gamma''-\beta''\gamma'=\gamma'\alpha''-\gamma''\alpha'=\alpha'\beta''-\alpha''\beta',$$
$$\omega+\varpi=\alpha'\alpha''+\beta'\beta''+\gamma'\gamma'':$$

the equation in question then is

$$(\rho-\omega)(\rho-\varpi)+\frac{Kl}{4\alpha\beta\gamma m}=0;$$

so that, calling the roots of it ρ_1, ρ_2, we have

$$\rho_1+\rho_2=\omega+\varpi,\quad \rho_1\rho_2=\omega\varpi+\frac{Kl}{4\alpha\beta\gamma m};$$

or we may write

$$\rho_1=\tfrac{1}{2}(\omega+\varpi+\sqrt{\Omega}),\ =\tfrac{1}{2}(\alpha'\alpha''+\beta'\beta''+\gamma'\gamma''+\sqrt{\Omega}),$$
$$\rho_2=\tfrac{1}{2}(\omega+\varpi-\sqrt{\Omega}),\ =\tfrac{1}{2}(\alpha'\alpha''+\beta'\beta''+\gamma'\gamma''-\sqrt{\Omega}),$$

if for shortness

$$\Omega=(\omega-\varpi)^2-\frac{Kl}{\alpha\beta\gamma m}.$$

162. I write also for shortness

$$a=(\beta-\gamma)\alpha'\alpha''+\alpha(\beta'\beta''-\gamma'\gamma''),\quad b=(\gamma-\alpha)\beta'\beta''+\beta(\gamma'\gamma''-\alpha'\alpha''),$$
$$c=(\alpha-\beta)\gamma'\gamma''+\gamma(\alpha'\alpha''-\beta'\beta'');$$

and I say that the singular planes are

(1) (1) $x=0,$

(2) (2) $y=0,$

(3) (3) $z=0,$

(4) (9) $$-\frac{1}{\beta\gamma}(X'-w)+(\rho_1-\omega)\left(\frac{y}{\beta}+\frac{z}{\gamma}\right)=0,$$

(5) (10) $$-\frac{1}{\gamma\alpha}(Y'-w)+(\rho_1-\omega)\left(\frac{z}{\gamma}+\frac{x}{\alpha}\right)=0,$$

(6) (11) $$-\frac{1}{\alpha\beta}(Z'-w)+(\rho_1-\omega)\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)=0,$$

(7) (13) $$-\frac{1}{\beta\gamma}(X''-w)+(\rho_2-\varpi)\left(\frac{y}{\beta}+\frac{z}{\gamma}\right)=0,$$

(8) (14) $$-\frac{1}{\gamma\alpha}(Y''-w)+(\rho_2-\varpi)\left(\frac{z}{\gamma}+\frac{x}{\alpha}\right)=0,$$

(9) (15) $$-\frac{1}{\alpha\beta}(Z''-w)+(\rho_2-\varpi)\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)=0,$$

(10) (8) $$P=0;$$

and that the nodes are

(1) (1) $(0, -\beta, \gamma, \alpha'\alpha''\beta\gamma)$,

(2) (2) $(\alpha, 0, -\gamma, \beta'\beta''\gamma\alpha)$,

(3) (3) $(-\alpha, \beta, 0, \gamma'\gamma''\alpha\beta)$,

(4) (9) $\{0, \rho_1-\gamma'\gamma'', -(\rho_2-\beta'\beta''), \alpha(\rho_1-\gamma'\gamma'')(\rho_2-\beta'\beta'')\}$,

(5) (10) $\{-(\rho_2-\gamma'\gamma''), 0, \rho_1-\alpha'\alpha'', \beta(\rho_1-\alpha'\alpha'')(\rho_2-\gamma'\gamma'')\}$,

(6) (11) $\{\rho_1-\beta'\beta'', -(\rho_2-\alpha'\alpha''), 0, \gamma(\rho_1-\beta'\beta'')(\rho_2-\alpha'\alpha'')\}$,

(7) (13) $\{0, \rho_2-\gamma'\gamma'', -(\rho_1-\beta'\beta''), \alpha(\rho_2-\gamma'\gamma'')(\rho_1-\beta'\beta'')\}$,

(8) (14) $\{-(\rho_1-\gamma'\gamma''), 0, \rho_2-\alpha'\alpha'', \beta(\rho_2-\alpha'\alpha'')(\rho_1-\gamma'\gamma'')\}$,

(9) (15) $\{\rho_2-\beta'\beta'', -(\rho_1-\alpha'\alpha''), 0, \gamma(\rho_2-\beta'\beta'')(\rho_1-\alpha'\alpha'')\}$,

(10) (8) $(0, 0, 0, 1)$,

(11) (5) $(1, 0, 0, 0)$,

(12) (6) $(0, 1, 0, 0)$,

(13) (7) $(0, 0, 1, 0)$,

(14) (16) $\left(\frac{1}{2}\alpha(a-\alpha\sqrt{\Omega}), \frac{1}{2}\beta(b-\beta\sqrt{\Omega}), \frac{1}{2}\gamma(c-\gamma\sqrt{\Omega}), \frac{kl}{4m}\right)$,

(15) (12) $\left(\frac{1}{2}\alpha(a+\alpha\sqrt{\Omega}), \frac{1}{2}\beta(b+\beta\sqrt{\Omega}), \frac{1}{2}\gamma(c+\gamma\sqrt{\Omega}), \frac{kl}{4m}\right)$.

163. The small reference numbers are those used by Kummer. It is, I think, better to retain the reference numbers belonging to the case of the 16-nodal surface; viz., there are here given, large, 1, 2, 3, 8, 9, 10, 11, 13, 14, 15 for the planes, and 1, 2, 3, 5, 6, ... to 16 for the nodes. Belonging to each node (that is, with the node as vertex) there is a quadric cone passing through 8 other nodes; and each node lies (exclusively of the cone whose vertex it is) in 8 such cones. We have thus the following square diagram :—

NODES	1	2	3	5	6	7	8	9	10	11	12	13	14	15	16	CONES
1	*	C8	C8	9,13	C1	C1	1,8	1,9	C13	C13	8,13	1,13	C9	C9	8,9	1, 8, 9, 13 (6, 7)
2	C8	*	C8	C2	10,14	C2	2,8	C14	2,10	C14	8,14	C10	2,14	C10	8,10	2, 8, 13, 14 (5, 7)
3	C8	C8	*	C3	C3	11,15	3,8	C15	C15	3,11	8,15	C11	C11	3,15	8,11	3, 8, 11, 15 (5, 6)
5	9,13	C2	C3	*	C3	C2	2,3	C9	2,13	3,13	C13	C13	2,9	3,9	C9	2, 3, 9, 13 (4, 5)
6	C1	10,14	C3	C3	*	C1	1,3	1,14	C10	3,14	C14	1,10	C14	3,10	C10	1, 3, 10, 14 (4, 6)
7	C1	C2	11,15	C2	C1	*	1,2	1,15	2,15	C11	C15	1,11	2,11	C15	C11	1, 2, 11, 15 (4, 7)
8	1,8	2,8	3,8	2,3	1,3	1,2	*	C1	C2	C3	C8	C1	C2	C3	C8	1, 2, 3, 8 (12, 16)
9	1,9	C14	C15	C9	1,14	1,15	C1	*	C15	C14	14,15	C1	9,14	9,15	C9	1, 9, 14, 15 (5, 16)
10	C13	2,10	C15	2,13	C10	2,15	C2	C15	*	C13	13,15	10,13	C2	10,15	C10	2, 10, 13, 15 (6, 16)
11	C13	C14	3,11	3,13	3,14	C11	C3	C14	C13	*	13,14	11,13	11,14	C3	C11	3, 11, 13, 14 (7, 16)
12	8,13	8,14	8,15	C13	C14	C15	C8	14,15	13,15	13,14	*	C13	C14	C15	C8	8, 13, 14, 15 (4, 12)
13	1,13	C10	C11	C13	1,10	1,11	C1	C1	10,13	11,13	C13	*	C11	C10	10,11	1, 10, 11, 13 (5, 12)
14	C9	2,14	C11	2,9	C14	2,11	C2	9,14	C2	11,14	C14	C11	*	C9	9,11	2, 9, 11, 14 (6, 12)
15	C9	C10	3,15	3,9	3,10	C15	C3	9,15	10,15	C3	C15	C10	C9	*	9,10	3, 9, 10, 15 (7, 12)
16	8,9	8,10	8,11	C9	C10	C11	C8	C9	C10	C11	C8	10,11	9,11	9,10	*	8, 9, 10, 11 (4, 16)

164. The arrangement is the same as in the 16-nodal square diagram; only, in the right-hand margin, a bracket (6, 7) denotes that instead of the planes (6, 7) we have a quadric cone; which cones are, in the body of the table, denoted by C. Thus for the node 1 the sextic cone is made up of the planes 1, 8, 9, 13 and of a quadric cone (6, 7), $=C$: the remaining 14 nodes lie on the nodal lines of the sextic cone, viz., the node 2 on an intersection of the cone C with the plane 8, the node 3 on an intersection of the same cone and plane, the node 5 on the intersection of the planes 9, 13, and so on.

The Equation of the 15*-nodal Surface, as deduced from that of the* 13 (*a*)*-nodal.*

165. If, in the equation hereafter given for the 13-nodal surface, we write $\nu=0$, $\mu=0$, or (what is the same thing) in that of the 14-nodal surface we write $\mu=0$, the form is

$$\begin{aligned} & w^2(A+4\lambda yz) \\ & +2w\,(B-2\lambda yzX) \\ & +\quad C=0. \end{aligned}$$

The circumscribed sextic cone is thus (2, 1, 1, 1, 1),

$$(\lambda\alpha^2yz+\beta\gamma x^2+\gamma\alpha xy+\alpha\beta xz)\,yz\,(P'-\rho_1P'')\,(P'-\rho_2P'')=0,$$

where ρ_1, ρ_2 now denote the roots of the equation

$$\lambda\,(\rho\alpha-\alpha'')^2-\frac{1}{\beta'\beta''\gamma'\gamma''}\,(\omega-\varpi)^2\,\rho\alpha'\alpha''=0.$$

The singular planes are

$$w=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

$$P'-\rho_1P''=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12),$$

$$P'-\rho_2P''=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16),$$

$$z=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$y=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\,\rho_1\,(Z'-w)-(Z''-w)=0 \ldots\ldots \quad (15),$$

$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\,\rho_2\,(Z'-w)-(Z''-w)=0 \ldots\ldots \quad (11),$$

$$\frac{\beta''\gamma'\alpha'}{\beta'\gamma''\alpha''}\,\rho_1\,(Y'-w)-(Y''-w)=0 \ldots\ldots \quad (14),$$

$$\frac{\beta''\gamma'\alpha'}{\beta'\gamma''\alpha''}\,\rho_2\,(Y'-w)-(Y''-w)=0 \ldots\ldots \quad (10),$$

$$X-w=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5);$$

and we have then the same square table as before: the coordinates of the 15 nodes may be obtained without difficulty.

166. The form is really equivalent to that first considered in regard to the 15-nodal surface. To show that this is so, we have only to arrange according to powers of x; viz., the equation thus becomes

$$x^2\{(\gamma\gamma'\gamma'')^2y^2+(\beta\beta'\beta'')^2z^2+w^2-2\beta\beta'\beta''zw+2\gamma\gamma'\gamma''wy+2\beta\beta'\beta''\gamma\gamma'\gamma''yz\}$$
$$+2x\{\gamma\gamma'\gamma''\alpha\alpha'\alpha''y^2z+\alpha\alpha'\alpha''\beta\beta'\beta''yz^2+\beta\beta'\beta''z^2w$$
$$-zw^2-w^2y-\gamma\gamma'\gamma''wy^2+\mathrm{M}yzw\}$$
$$+(\alpha\alpha'\alpha''yz+wy-wz)^2-4\lambda yzw(\mathrm{X}-w)=0,$$

where, if for a moment A denotes the coefficient of x^2, we have $y=0$, $z=0$, $w=0$, $\mathrm{X}-w=0$, four tangent planes of the quadric cone $\mathrm{A}=0$.

14-*nodal Surface* $8(3_1,1,1,1)+6(2,2,1,1)$,
Node-form $(3_1,1,1,1)$.

167. In the equation hereafter given for the 13-nodal surface, writing $\nu=0$, the sextic cone becomes

$$4z(\lambda\alpha^2y^2z+\mu\beta^2zx^2+\beta\gamma x^2y+\gamma\alpha xy^2+\alpha\beta xyz)(\mathrm{P}'-\rho_1\mathrm{P}'')(\mathrm{P}'-\rho_2\mathrm{P}'')=0;$$

viz., this is of the form in question $(3_1,1,1,1)$; and the equation of the surface is

$$w^2\mathrm{A}+4(\lambda y+\mu x)z-4\lambda\mu z^2$$
$$+2w\{\mathrm{B}-2(\lambda y\mathrm{X}+\mu x\mathrm{Y})z\}$$
$$+\mathrm{C}=0.$$

The singular planes are

$w=0$ ..... (4) ... (4)

$\mathrm{P}'-\rho_1\mathrm{P}''=0$ ..... (12) ... (2)

$\mathrm{P}'-\rho_2\mathrm{P}''=0$ ..... (16) ... (3)

$z=0$ ..... (3) ... (1)

$\dfrac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_1(\mathrm{Z}'-w)-(\mathrm{Z}''-w)=0$ ..... (15) ... (6)

$\dfrac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_2(\mathrm{Z}'-w)-(\mathrm{Z}''-w)=0$ ..... (11) ... (5)

and the nodes are

$(0,0,0,1)$ ..... (8) ... (1)

$(1,0,0,0)$ ..... (5) ... (13)

$(0,1,0,0)$ ..... (6) ... (14)

$(0,0,1,0)$ ..... (7) ... (5)

$\left(-\dfrac{\beta''-\beta'\rho_1}{\beta'\beta''},\ \dfrac{\alpha''-\alpha'\rho_1}{\alpha'\alpha''},\ 0,\ \dfrac{\gamma\gamma'\gamma''(\beta''-\beta'\rho_1)(\alpha''-\alpha'\rho_1)}{\alpha''\beta''\gamma'-\alpha'\beta'\gamma''\rho_1}\right)$ ..... (11) ... (4

$\left(-\dfrac{\beta''-\beta'\rho_2}{\beta'\beta''},\ \dfrac{\alpha''-\alpha'\rho_2}{\alpha'\alpha''},\ 0,\ \dfrac{\gamma\gamma'\gamma''(\beta''-\beta'\rho_2)(\alpha''-\alpha'\rho_2)}{\alpha''\beta''\gamma'-\alpha'\beta'\gamma''\rho_2}\right)$ ..... (15) ... 3

$$\left(\frac{\alpha\alpha'\alpha''}{\alpha''-\alpha'\rho_1}, \frac{\beta\beta'\beta''}{\beta''-\beta'\rho_1}, \frac{\gamma\gamma'\gamma''}{\gamma''-\gamma'\rho_1}, 0\right) \ldots\ldots\ldots (16) \ldots (7)$$

$$\left(\frac{\alpha\alpha'\alpha''}{\alpha''-\alpha'\rho_2}, \frac{\beta\beta'\beta''}{\beta''-\beta'\rho_2}, \frac{\gamma\gamma'\gamma''}{\gamma''-\gamma'\rho_2}, 0\right) \ldots\ldots\ldots (12) \ldots (8)$$

$$(\alpha'\alpha'', \beta'\beta', \gamma'\gamma'', 0) \ldots\ldots\ldots (4) \ldots (6)$$

$$(-\alpha, \beta, 0, \gamma'\gamma''\alpha\beta) \ldots\ldots\ldots (3) \ldots (2)$$

two nodes on line

$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_2 (Z-w)-(Z''-w)=0, \quad P'-\rho_1 P''=0 \ldots\ldots (13,14) \ldots (9,10)$$

and two nodes on line

$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_1 (Z'-w)-(Z''-w)=0, \quad P'-\rho_2 P''=0 \ldots\ldots (9,10) \ldots (11,12)$$

where the large numbers are those for the 16-nodal surface, the small numbers are Kummer's.

168. In these formulæ, ρ_1, ρ_2 denote the roots of the equation

$$\begin{aligned}&(\lambda\beta''^2+\mu\alpha''^2)(\rho\alpha'\beta')^2\\ &+(\lambda\beta'^2+\mu\alpha'^2)(\alpha''\beta'')^2\\ &-2\left[\lambda\beta'\beta''+\mu\alpha'\alpha''+\frac{1}{2\gamma'\gamma''}(\omega-\varpi)^2\right]\rho\alpha'\beta'\alpha''\beta''=0.\end{aligned}$$

169. The relation of the nodes and sextic cones is given by means of the square diagram—

NODES	3	4	5	6	7	8	9	10	11	12	13	14	15	16	CONES
3	*	×	C3	C3	11,15	C3	C15	C15	3,11	C15	C11	C11	3,15	C11	3, 11, 15, (5, 6, 8)
4	×	*	C4	C4	C4	12,16	C16	C16	C16	4,12	C12	C12	C12	4,16	4, 12, 16, (5, 6, 7)
5	D3	D4	*	3,4	D4	D3			D'3	D'4			D'3	D'4	3, 4, (2, 5), (9, 13)
6	D3	D4	3,4	*	D4	D3			D'3	D'4			D'3	D'4	3, 4, (1, 6), (10, 14)
7	11,15	C4	C4	C4	*	×	C15	C15	C11	4,15	C11	C11	C15	4,11	4, 11, 15, (1, 2, 7)
8	C3	12,16	C3	C3	×	*	C16	C16	3,16	C12	C12	C12	3,12	C16	3, 12, 16, (1, 2, 8)
9	D15	D16			D'15	D'16	*	15,16	D16	D15			D'15	D'16	15, 16, (5, 14), (1, 9)
10	D15	D16			D'15	D'16	15,16	*	D16	D15			D'15	D'16	15, 16, (6, 13), (2, 10)
11	3,11	C16	C3	C3	C11	3,16	C16	C16	*	×	C11	C11	C3	11,16	3, 11, 16, (7, 13, 14)
12	C15	4,12	C4	C4	4,15	C12	C15	C15	×	*	C12	C12	12,15	C4	4, 12, 15, (8, 13, 14)
13	D11	D12			D'11	D'12			D'11	D'12	*	11,12	D12	D11	11, 12, (5, 10), (1, 13)
14	D11	D12			D'11	D'12			D'11	D'12	11,12	*	D12	D11	11, 12, (6, 9), (2, 14)
15	3,15	C12	C3	C3	C15	3,12	C15	C15	C3	12,15	C12	C12	*	×	3, 12, 15, (7, 9, 10)
16	C11	4,16	C4	C4	4,11	C16	C16	C16	11,16	C4	C11	C11	×	*	4, 11, 16, (8, 9, 10)

where the arrangement is the same as before; only in the right-hand margin, a bracket (5, 6, 8) denotes that, instead of the planes 5, 6, 8, we have a nodal cubic cone (5, 6, 8), which is in the body of the table referred to as C, and the nodal line thereof by $\times$; and the brackets (2, 5), (9, 13) denote that, instead of the planes 2, 5, we have a quadric cone (2, 5), and instead of the planes 9, 13, a quadric cone (9, 13); which cones are in the body of the table referred to as D, D′ respectively. And it is to be understood that each vacant square of the table should contain the symbol (D, D′), this being omitted only for the purpose of better exhibiting the form of the table.

170. The equation of the surface may be presented in the irrational form

$$\sqrt{(\mathrm{P}'-\rho_1\mathrm{P}'')[\gamma''\alpha'\beta'\rho_2(\mathrm{Z}'-w)-\gamma'\alpha''\beta''(\mathrm{Z}''-w)]}$$
$$+\sqrt{(\mathrm{P}'-\rho_2\mathrm{P}'')[\gamma''\alpha'\beta'\rho_1(\mathrm{Z}'-w)-\gamma'\alpha''\beta''(\mathrm{Z}''-w)]}$$
$$+(\rho_1-\rho_2)\sqrt{\alpha'\beta'\alpha''\beta''\left(\frac{\lambda}{\alpha''^2}+\frac{\mu}{\beta''^2}\right)}zw=0.$$

In fact the norm of the left-hand side is

$$=(\rho_1-\rho_2)^2(\omega-\varpi)^2\{w^2[\mathrm{A}+4z(\lambda y+\mu x)-4\lambda\mu z^2]$$
$$+2w[\mathrm{B}-2z(\lambda y\mathrm{X}+\mu x\mathrm{Y})]$$
$$+\mathrm{C}\qquad\qquad\}.$$

To partially verify this, observe that, writing the equation under the form $\sqrt{\mathrm{R}}+\sqrt{\mathrm{S}}+\sqrt{\mathrm{T}}=0$, on writing therein $w=0$, we ought to have

$$(\mathrm{R}-\mathrm{S})^2=(\rho_1-\rho_2)^2(\omega-\varpi)^2(\alpha\alpha'\alpha''yz+\beta\beta'\beta''zx+\gamma\gamma'\gamma''xy)^2.$$

But writing $w=0$, we have

$$\begin{aligned}\mathrm{R}-\mathrm{S}=&\quad(\mathrm{P}'-\rho_1\mathrm{P}'')(\gamma''\alpha'\beta'\rho_2\mathrm{Z}'-\gamma'\alpha''\beta''\mathrm{Z}'')\\&-(\mathrm{P}'-\rho_2\mathrm{P}'')(\gamma''\alpha'\beta'\rho_1\mathrm{Z}'-\gamma'\alpha''\beta''\mathrm{Z}'')\\=&\,(\rho_2-\rho_1)(\gamma''\alpha'\beta'\mathrm{P}'\mathrm{Z}'-\gamma'\alpha''\beta''\mathrm{P}''\mathrm{Z}'')\\=&\,(\rho_2-\rho_1)\{\gamma''(\beta'\gamma'x+\gamma'\alpha'y+\alpha'\beta'z)(\beta\beta''x-\alpha\alpha''y)\\&\qquad-\gamma'(\beta''\gamma''x+\gamma''\alpha''y+\alpha''\beta''z)(\beta\beta'x-\alpha\alpha'y)\},\end{aligned}$$

which is easily found to be

$$=-(\rho_2-\rho_1)(\omega-\varpi)(\alpha\alpha'\alpha''yz+\beta\beta'\beta''zx+\gamma\gamma'\gamma''xy);$$

and thus $(\mathrm{R}-\mathrm{S})^2$ has the value in question.

171. In further verification, observe that, writing $x,y,z=\alpha'\alpha'',\beta'\beta'',\gamma'\gamma''$, and therefore $\mathrm{P}'=0$, $\mathrm{P}''=0$, we ought to have

$$(\rho_1-\rho_2)^2(\alpha'\beta'\alpha''\beta'')^2\left(\frac{\lambda}{\alpha''^2}+\frac{\mu}{\beta''^2}\right)z^2w^2$$
$$=(\rho_1-\rho_2)^2(\omega-\varpi)^2w^2[\mathrm{A}+4z(\lambda y+\mu x)-4\lambda\mu z^2)],$$

observing that, for the values in question, B, X, Y, C all vanish.

This is

$$(\rho_1-\rho_2)^2(\alpha'\beta'\alpha''\beta'')^2\left(\frac{\lambda}{\alpha''^2}+\frac{\mu}{\beta''^2}\right)^2(\gamma'\gamma'')^2$$
$$=(\omega-\varpi)^2\{(\omega-\varpi)^2+4\gamma'\gamma''(\lambda\beta'\beta''+\mu\alpha'\alpha''-\lambda\mu\gamma'\gamma'')\},$$

which is in fact the value of $(\rho_1-\rho_2)^2$ obtained from the equation in ρ.

The $13(\alpha)$*-nodal Surface* $3(4_3,1,1)+1(3,1,1,1)+9(3_1,2,1)$.

172. The equation, node-form $(4_3,1,1)$, is

$$w^2\{A+4(\lambda yz+\mu zx+\nu xy)-4(\mu\nu x^2+\nu\lambda y^2+\lambda\mu z^2)\}$$
$$+2w\{B-2(\lambda yzX+\mu zxY+\nu xyZ)\}$$
$$+\quad C=0;$$

viz., for the circumscribed cone we have

$$\{A+4(\lambda yz+\mu zx+\nu xy)-4(\mu\nu x^2+\nu\lambda y^2+\lambda\mu z^2)\}C$$
$$-\{B-2(\lambda yzX+\mu zxY+\nu xyZ)\}^2$$
$$=4\{\lambda\alpha^2y^2z^2+\mu\beta^2z^2x^2+\nu\gamma^2x^2y^2+\beta\gamma x^2yz+\gamma\alpha y^2zx+\alpha\beta z^2xy\}$$
$$\times\left\{-\lambda\frac{X^2}{\alpha^2}-\mu\frac{Y^2}{\beta^2}-\nu\frac{Z^2}{\gamma^2}+\alpha'\alpha''\beta'\beta''\gamma'\gamma''P'P''\right\},$$

where, on the right-hand side, the first factor, equated to zero, represents a trinodal quartic cone, the nodal lines whereof are $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$.

173. As regards the second factor, it is to be observed that, writing as above $\omega-\varpi=\beta\gamma'-\beta''\gamma'$, $=\gamma'\alpha''-\gamma''\alpha'$, $=\alpha'\beta''-\alpha''\beta'$, we have identically

$$\alpha'P''-\alpha''P''=\frac{\omega-\varpi}{\beta'\beta''\gamma'\gamma''}\frac{X}{\alpha},$$
$$\beta'P'-\beta''P''=\frac{\omega-\varpi}{\gamma'\gamma''\alpha'\alpha''}\frac{Y}{\beta},$$
$$\gamma'P'-\gamma''P''=\frac{\omega-\varpi}{\alpha'\alpha''\beta'\beta''}\frac{Z}{\gamma};$$

so that the second factor is

$$\frac{1}{(\omega-\varpi)^2}\{-\lambda(\beta'\beta''\gamma'\gamma'')^2(\alpha'P''-\alpha''P'')^2-\mu(\gamma'\gamma''\alpha'\alpha'')^2(\beta'P'-\beta''P'')^2$$
$$-\nu(\alpha'\alpha''\beta'\beta'')^2(\gamma'P'-\gamma''P'')^2+(\omega-\varpi)^2\alpha'\alpha''\beta'\beta''\gamma'\gamma''P'P''\};$$

or what is the same thing,

$$\frac{1}{(\omega-\varpi)^2}\{-[\lambda(\beta''\gamma'')^2+\mu(\gamma''\alpha'')^2+\nu(\alpha''\beta'')^2](\alpha'\beta'\gamma'P')^2$$
$$-[\lambda(\beta'\gamma')^2+\mu(\gamma'\alpha')^2+\nu(\alpha'\beta')^2](\alpha''\beta''\gamma'P'')^2$$
$$+2[(\lambda\beta'\gamma'\beta''\gamma'')+\mu(\gamma'\alpha'\gamma''\alpha'')+\nu(\alpha'\beta'\alpha''\beta'')+\tfrac{1}{2}(\omega-\varpi)^2]$$
$$\times\alpha'\beta'\gamma'P'\ \alpha''\beta''\gamma''P''\}.$$

so that, equating to zero, we have a pair of planes, each passing through the line $P'=0$, $P''=0$, and which it is clear must be the tangent planes from this line to the quadric cone $A'=0$. I will presently return to this, but consider first the foregoing identical equation in regard to the circumscribed cone.

174. In verification hereof, observe first that, if $x=0$, the equation becomes

$$\begin{aligned}\{(y-z)^2+4\lambda yz-4\lambda(\nu y^2+\mu z^2)\}\,(\alpha\alpha'\alpha'')^2y^2z^2\\ -\{\alpha\alpha'\alpha''(y^2z-yz^2)-2\lambda yz\alpha(\gamma'\gamma''y-\beta'\beta''z)\}^2\\ =4\lambda y^2z^2\{-\lambda\alpha^2(\gamma'\gamma''y-\beta'\beta''z)^2-(\alpha\alpha'\alpha'')^2(\nu y^2+\mu z^2)\\ -\alpha^2\alpha'\alpha''(\gamma'y+\beta'z)(\gamma''y+\beta''z)\};\end{aligned}$$

viz., omitting terms which destroy each other, this is

$$\begin{aligned}[(y-z)^2+4\lambda yz](\alpha\alpha'\alpha'')^2y^2z^2-[\alpha\alpha'\alpha''(y^2z-yz^2)-2\lambda yz\alpha(\gamma'\gamma''y-\beta'\beta''z)]^2\\ =4\lambda y^2z^2[-\lambda\alpha^2(\gamma'\gamma''y-\beta'\beta''z)^2-\alpha^2\alpha'\alpha''(\gamma'y+\beta'z)(\gamma''y+\beta''z)].\end{aligned}$$

Or again, this is

$$\begin{aligned}4\lambda yz\,(\alpha\alpha'\alpha'')^2y^2z^2+4\lambda yz\,\alpha^2\alpha'\alpha''(y^2z-yz^2)(\gamma'\gamma''y-\beta'\beta'')z\\ =-4\lambda y^2z^2\,\alpha^2\alpha'\alpha''(\gamma'y+\beta'z)(\gamma''y+\beta''z);\end{aligned}$$

viz., this is

$$\alpha'\alpha''yz+(y-z)(\gamma'\gamma''y-\beta'\beta''z)=(\gamma'y+\beta'z)(\gamma''y+\beta''z),$$

which is at once seen to be true.

175. Again, compare the terms which contain x^4yz. On the right hand side, we have

$$4\beta y\,x^3yz\times\text{term in }x^2\text{ of}$$

$$\left(-\lambda\frac{X^2}{\alpha^2}-\mu\frac{Y^2}{\beta^2}-\nu\frac{Z^2}{\gamma^2}+\alpha'\alpha''\beta'\beta''\gamma'\gamma''P'P''\right);$$

viz., the coefficient is

$$=4\beta\gamma\{-\mu(\gamma'\gamma'')^2-\nu(\beta'\beta'')^2+\beta'\beta''\gamma'\gamma''\}.$$

On the left hand side, that is in $A'C-B'^2$, the only terms which give rise to the terms in question are

$$\text{in A', }(1-4\mu\nu)x^2;\quad\text{in C, }2\beta\beta'\beta''\gamma\gamma'\gamma''x^3yz;$$

and in B' $\quad(\gamma\gamma'\gamma''-2\nu\gamma\beta'\beta'')x^2y,\quad-(\beta\beta'\beta''-2\mu\beta\gamma'\gamma'')x^2z;$

whence the coefficient is

$$\begin{aligned}2\beta\beta'\beta''\gamma\gamma'\gamma''(1-4\mu\nu)\\ +2(\gamma\gamma'\gamma''-2\nu\gamma\beta'\beta'')(\beta\beta'\beta''-2\mu\beta\gamma'\gamma''),\end{aligned}$$

which is in fact

$$=4\beta\gamma\{-\mu(\gamma'\gamma'')^2-\nu(\beta'\beta'')^2+\beta'\beta''\gamma'\gamma''\},$$

which is right; and the verification may be completed without difficulty.

176. The singular planes are

$$w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$
$$P' - \rho_1 P'' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$
$$P' - \rho_2 P'' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (16);$$

and the nodes are

$$(0,\ 0,\ 0,\ 1) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8) \quad \ldots\ (1)$$
$$(1,\ 0,\ 0,\ 0) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5) \quad \ldots\ (5)$$
$$(0,\ 1,\ 0,\ 0) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6) \quad \ldots\ (6)$$
$$(0,\ 0,\ 1,\ 0) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7) \quad \ldots\ (7)$$
$$\left(\frac{\alpha\alpha'\alpha''}{\alpha'' - \alpha'\rho_1},\ \frac{\beta\beta'\beta''}{\beta'' - \beta'\rho_1},\ \frac{\gamma\gamma'\gamma''}{\gamma'' - \gamma'\rho_1},\ 0\right) \quad \ldots\ldots\ldots \quad (16) \quad \ldots\ (2)$$
$$\left(\frac{\alpha\alpha'\alpha''}{\alpha'' - \alpha'\rho_2},\ \frac{\beta\beta'\beta''}{\beta'' - \beta'\rho_2},\ \frac{\gamma\gamma'\gamma''}{\gamma'' - \gamma'\rho},\ 0\right) \quad \ldots\ldots\ldots \quad (12) \quad \ldots\ (3)$$
$$(\ \alpha'\alpha''\ ,\ \ \beta'\beta''\ ,\ \ \gamma'\gamma''\ ,\ 0) \quad \ldots\ldots\ldots \quad (4) \quad \ldots\ (4)$$

three nodes in $P' - \rho_1 P'' = 0$ ..... (13, 14, 15) ... (8, 9, 10)

„ „ $P' - \rho_2 P'' = 0$ ..... (9, 10, 11) ... (11, 12, 13);

where the small numbers are those used by Kummer, the large ones are those referring to the 16-nodal surface, and are here adopted. In the foregoing formulæ ρ_1, ρ_2 are the roots of

$$[\lambda(\beta''\gamma'')^2 + \mu(\gamma''\alpha'')^2 + \nu(\alpha''\beta'')^2](\rho\alpha'\beta'\gamma')^2$$
$$+ [\lambda(\beta'\gamma')^2 + \mu(\gamma'\alpha')^2 + \nu(\alpha'\beta')^2](\alpha''\beta''\gamma'')^2$$
$$- 2[\lambda\beta'\gamma'\beta''\gamma'' + \mu\gamma'\alpha'\gamma''\alpha'' + \nu\alpha'\beta'\alpha''\beta'' + \tfrac{1}{2}(\omega - \varpi)^2]\rho\alpha'\beta'\gamma'\alpha''\beta''\gamma'' = 0.$$

177. We have the square diagram

NODES	4	5	6	7	8	9	10	11	12	13	14	15	16	CONES
4	*	C4	C4	C4	12,16	C16	C16	C16	4,12	C12	C12	C12	4,16	4, 12, 16, (5, 6, 7)
5	C4	*	C4	C4	×				C′4				C′4	4 (2, 3 ; 5) nodal, (9, 13)
6	C4	C4	*	C4	×				C′4				C′4	4 (1, 3 ; 6) nodal, (10, 14)
7	C4	C4	C4	*	×				C′4				C′4	4 (1, 2 ; 7) nodal, (11, 15)
8	12,16	×	×	×	*	C16	C16	C16	C16	C12	C12	C12	C12	12, 16 (8 ; 1, 2, 3) trinodal
9	C16				C′16	*	C16	C16	×				C′16	16, (5 ; 14, 15) nodal, (1, 9)
10	C16				C′16	C16	*	C16	×				C′16	16, (6 ; 13, 15) nodal, (2, 10)
11	C16				C′16	C16	C16	*	×				C′16	16, (7 ; 13, 14) nodal, (3, 11)
12	4,12	C4	C4	C4	C12	×	×	×	*	C12	C12	C12	C4	4, 12 (8 ; 13, 14, 15) trinodal
13	C12				C′12				C′12	*	C12	C12	×	12, (5 ; 10, 11) nodal, (1, 13)
14	C12				C′12				C′12	C12	*	C12	×	12, (6 ; 9, 11) nodal, (2, 14)
15	C12				C′12				C′12	C12	C12	*	×	12, (7 ; 9, 10) nodal, (3, 15)
16	4,16	C4	C4	C4	C16	C16	C16	C16	C4	×	×	×	*	4, 16 (8 ; 9, 10, 11) trinodal

where for greater clearness I have omitted the symbol CC′, which is to be understood as occupying each of the vacant squares.

The arrangement is the same as before; the right-hand margin shows the sextic cone; viz., for the node 4 this is made up of the singular planes 4, 12, 16 and of a cubic cone represented by (5, 6, 7) (as replacing the planes 5, 6, 7 in the case of the 16-nodal surface). Similarly for the node 5, the sextic cone is made up of the singular plane 4, the nodal cubic cone (2, 3 ; 5), and the quadric cone (9, 13) (the numbers in these last symbols indicating the planes in the case of the 16-nodal surface, which are here replaced by cones). So for the node 8, the sextic cone is made up of the singular planes 12, 16 and of the trinodal quartic cone (8 ; 1, 2, 3). As regards a nodal cubic cone, for example (2, 3 ; 5), the semicolon is used to indicate that the nodal line replaces the intersection of the planes 2, 3 ; the other intersections 2, 5 and 3, 5 having disappeared. And so for a trinodal quartic cone (8 ; 1, 2, 3), the semicolon is used to indicate that the nodal lines replace the intersection (1, 2), the intersection (1, 3), and the intersection (2, 3) respectively ; the other intersections 1, 2 ; 2, 8 and 3, 8 having disappeared. Finally, in the body of the table, C is used to denote the cubic or the quartic cone (as the case may be) ; × to denote a nodal line of either of these cones ; and C′ the quadric cone ; as already mentioned, the vacant places are considered to be CC′.

The reading of the table is then as follows ; viz., for the node 4, the remaining twelve nodes lie on the nodal lines of the sextic cone 4, 12, 16, (5, 6, 7), as shown ; viz., 5, 6, 7 are each of them on the intersection of the cubic cone with the plane 4 ; 8 is on the intersection of the planes 12 and 16 ; and so on.

I reserve for another Memoir the discussion of the 13 (β)-nodal surface, and the surfaces with less than 13 nodes.

Presents received :—

" Monatsbericht," Nov., Dec., 1870 ; Jan., 1871.

" Proceedings of the Royal Society," No. 126.

" Journal of the London Institution," (4).

May 11*th*, 1871.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Mr. C. J. Monro, B.A., late Fellow of Trinity College, Cambridge, was elected a Member. Mr. J. Griffiths, M.A., Fellow of Jesus College, Oxford, was proposed for election ; and the Hon. J. W. Strutt, M.A., Fellow of Trinity College, Cambridge, was admitted into the Society.

Prof. Henrici, V.P., indicated the method of treatment he had employed in his paper "On the Singularities of the Envelopes of a Non-Unicursal Series of Curves." Prof. Cayley made a few remarks upon the paper.

The Hon. J. W. Strutt then read the following paper:—

Note on the Explanation of Coronas, as given in Verdet's "Leçons d'Optique physique," and other Works.

Coronas are formed whenever, between the eye and the source of light, a large number of small opaque bodies, regular in size but irregular in distribution, are to be found. By a principle known as Babinet's, it is permitted to substitute for the layer of obstructing globules an opaque screen having apertures, which admit the light precisely where before it was stopped. From each aperture diverges a secondary wave, whose phase is arbitrarily related to those of the other waves on account of the irregularity in the distribution of the centres from which they emanate. In the theory of coronas, the intensities of the partial waves are supposed to be equal.

Any vibration may be represented by a straight line drawn from a given point, whose length is proportional to the amplitude, and angular position depends on the phase. When several vibrations affect the same point, the resultant may be found from the representative lines, according to the ordinary rules for compounding directed magnitudes. We have, therefore, to consider the resultant of a great number of lines radiating from an origin whose lengths are all equal and directions thoroughly irregular. From this statement, it appears at once that the question is perfectly indeterminate, and that no conclusion whatever can be drawn as to the probable value of the phase of the resultant, while all that could be inferred as to the amplitude relates merely to the order of its magnitude. Nevertheless, Verdet (vol. i., p. 297) comes to the conclusion that, the number of partial vibrations being n, the resultant is definitely $\sqrt{n}$ times greater than each component, provided, of course, that n is a large number. As to its phase, he is silent; and, if asked the question, would, I think, have found it difficult to explain how a resultant vibration could have a definite amplitude in conjunction with an indeterminate phase. I propose to point out what I conceive to be the error which vitiates his reasoning, and afterwards to show that, after all, his theory is substantially correct.

The component vibrations may be expressed by

$$\sin 2\pi\left(\frac{t}{\tau}\right),\ \sin 2\pi\left(\frac{t}{\tau}+\frac{\delta_1}{\lambda}\right),\ \sin 2\pi\left(\frac{t}{\tau}+\frac{\delta_2}{\lambda}\right),\ \ldots\ldots\ldots$$

The intensity of the light corresponding to the sum of these is

$$\left(1+\cos 2\pi\frac{\delta_1}{\lambda}+\cos 2\pi\frac{\delta_2}{\lambda}+\ldots\ldots\right)^2$$
$$+\left(\sin 2\pi\frac{\delta_1}{\lambda}+\sin 2\pi\frac{\delta_2}{\lambda}+\ldots\ldots\right)^2,$$

or
$$n+2\cos 2\pi\frac{\delta_1}{\lambda}+2\cos 2\pi\frac{\delta_2}{\lambda}+\ldots\ldots$$
$$\ldots\ldots+2\cos 2\pi\frac{\delta_1-\delta_2}{\lambda}+2\cos 2\pi\frac{\delta_1-\delta_3}{\lambda}+\ldots\ldots+2\cos 2\pi\frac{\delta_2-\delta_3}{\lambda}+\ldots\ldots$$

According to Verdet, this reduces to n when n is great enough, in consequence of the tendency of the series of cosines to zero. The inference is, I believe, unwarranted. All that we are entitled to say is, that when n is great, the sum of the cosines of the arbitrary angles tends to vanish, *in comparison with the number of terms of the series*, which is of the order n^2. There is no reason for supposing that the series becomes small in comparison with n, as the argument requires.* The analysis, as it here stands, is taken from Billet ("Traité d'Optique physique," tom. i., p. 235). In Verdet's book there is, in addition to the above fallacy, a mistake in the work, which I need not further refer to.

I conclude, therefore, that the effect of the diffraction of a plane wave coming through a great number of small holes in an opaque screen is not correctly given. Yet the error disappears in the application to the theory of coronas as they occur in Nature. For Verdet's result comes only to this, that the sources of light are to be treated as

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* It has been suggested to me that I have not sufficiently explained why the series of cosines does not tend to vanish when n is large enough. In order to examine this point more closely, let us, instead of comparing the phases of the second and succeeding vibrations with the first, refer them all to an external standard, so that the component vibrations are expressed by

$$\sin 2\pi\left(\frac{t}{\tau}+\frac{\delta_0}{\lambda}\right),\quad \sin 2\pi\left(\frac{t}{\tau}+\frac{\delta_1}{\lambda}\right),\quad \ldots\ldots\ldots$$

The expression for the resulting illumination will be

$$\left(\cos 2\pi\frac{\delta_0}{\lambda}+\cos 2\pi\frac{\delta_1}{\lambda}+\ldots\ldots\right)^2+\left(\sin 2\pi\frac{\delta_0}{\lambda}+\sin 2\pi\frac{\delta_1}{\lambda}+\ldots\ldots\right)^2.$$

The phases of the original vibrations being distributed with absolute irregularity, each series within brackets tends to zero; for the cosines or sines are, in the long run, as much positive as negative. The reason why the former series of cosines does not tend to vanish, but to the limit $-n$, is now evident. The reference of the phases of the succeeding components to the first as a standard has detracted something from the arbitrary character of the magnitudes which express them, so that the angles $2\pi\frac{\delta_1-\delta_2}{\lambda}$, $2\pi\frac{\delta_1-\delta_3}{\lambda}$, ... are no longer to be regarded as distributed absolutely at random over the entire circumference. I hope that this explanation, which is somewhat difficult to put into words, though the idea is simple enough, will be considered sufficient. At the same time, I submit that the burden of proof lay with those against whom I am arguing, when once the absurdity of the conclusion was pointed out.

independent, and the illuminations due to them, and not the vibrations, are to be compounded. As it is commonly though not very correctly expressed, there is to be no interference. For this the sensible apparent magnitude of the sun, or other source of light, is an amply sufficient cause.

At page 106 of his excellent lectures, Verdet shows that the disturbance from the sun cannot be considered as a system of plane waves over a space greater than a circle of $\frac{1}{2}\frac{1}{0}$ of a millimetre diameter. Between the vibrations at two points whose mutual distance is much greater than this, there is no permanent relation of phase. This shows that the vibrations corresponding to two holes in the imaginary screen cannot interfere regularly, but behave as if they were due to thoroughly independent sources of light.

Sir W. Thomson, Prof. Maxwell, and Mr. Strutt made some further remarks on the subject of the paper. Mr. Maxwell then gave a description of two singular solar halos which he had recently seen; and Prof. Adams, of King's College, gave some additional particulars in the case of one of the phenomena which had also been noticed by himself. Prof. Cayley, V.P., communicated an account of the following paper:—

On the Problem of finding the Circle which cuts three given Circles at three given Angles. By J. Griffiths, M.A.

I shall first show that a circle which cuts three given circles S_1, S_2, S_3 at the given angles θ_1, θ_2, θ_3, is always touched by a variable circle $\lambda S_1+\mu S_2+\nu S_3$, where λ, μ, ν are connected by a certain relation of the second degree.

To simplify our investigation, let us take the centre of the required circle, U say, as the origin of coordinates, so that, if R be its radius, we have for its equation $U = x^2+y^2-R^2 = 0$;

and for those of the given circles S_1, S_2, S_3, which are cut by U at angles θ_1, θ_2, θ_3 respectively,

$$S_1 = x^2+y^2+2g'x+2f'y+R^2-2Rr_1\cos\theta_1 = 0,$$
$$S_2 = x^2+y^2+2g''x+2f''y+R^2-2Rr_2\cos\theta_2 = 0,$$
$$S_3 = x^2+y^2+2g'''x+2f'''y+R^2-2Rr_3\cos\theta_3 = 0,$$

r_1, r_2, r_3 being the respective radii of S_1, S_2, S_3.

Now let us consider the circle $\lambda S_1+\mu S_2+\nu S_3$. Its equation may be written in the form

$$(\lambda+\mu+\nu)(x^2+y^2+R^2)+2\Sigma\lambda g'x+2\Sigma\lambda f'y-2R\Sigma(\lambda r_1\cos\theta_1) = 0,$$

where $\Sigma\lambda g' = \lambda g'+\mu g''+\nu g'''$, &c.;

and the condition that it shall *touch* U is

$$\{(\mathrm{R}-r_1\cos\theta_1)\lambda+(\mathrm{R}-r_2\cos\theta_2)\mu+(\mathrm{R}-r_3\cos\theta_3)\nu\}^2 = (g'\lambda+g''\mu+g'''\nu)^2+(f'\lambda+f''\mu+f'''\nu)^2,$$

or $$(l^2-\delta'^2)\lambda^2+(m^2-\delta''^2)\mu^2+(n^2-\delta'''^2)\nu^2 + 2(mn-g''g'''-f''f''')\mu\nu+\ldots\ldots+\ldots\ldots=0,$$

if we write l, m, n for $\mathrm{R}-r_1\cos\theta_1$, $\mathrm{R}-r_2\cos\theta_2$, $\mathrm{R}-r_3\cos\theta_3$ respectively, and δ', δ'', δ''' for $\sqrt{g'^2+f'^2}$, $\sqrt{g''^2+f''^2}$, $\sqrt{g'''^2+f'''^2}$.

Observing, then, that δ', δ'', δ''' must be the distances between the centre of the required circle U and those of S_1, S_2, S_3, we have

$$\sqrt{\delta'^2-r_1^2\sin^2\theta_1}=\mathrm{R}-r_1\cos\theta_1=l,$$
$$\sqrt{\delta''^2-r_2^2\sin^2\theta_2}=\mathrm{R}-r_2\cos\theta_2=m,$$
$$\sqrt{\delta'''^2-r_3^2\sin^2\theta_3}=\mathrm{R}-r_3\cos\theta_3=n;$$

since, evidently, by the conditions of the problem,

$$\delta'^2=\mathrm{R}^2+r_1^2-2\mathrm{R}r_1\cos\theta_1,\quad\&c.$$

The coefficient of λ^2, for instance, in the above equation of condition, therefore becomes $\quad l^2-\delta'^2=-r_1^2\sin^2\theta_1$;

and that of $\mu\nu$,

$$\begin{aligned}2(mn-g''g'''-f''f''') &= -(m-n)^2+(g''-g''')^2+(f''-f''')^2+m^2+n^2-g''^2-g'''^2-f''^2-f'''^2\\ &= -(r_2\cos\theta_2-r_3\cos\theta_3)^2+\delta_1^2+\delta''^2-r_2^2\sin^2\theta_2+\delta'''^2-r_3^2\sin^2\theta_3-\delta''^2-\delta'''^2\\ &= \delta_1^2-r_2^2-r_3^2+2r_2r_3\cos\theta_2\cos\theta_3,\end{aligned}$$

where $\delta_1^2=(g''-g''')^2+(f''-f''')^2=$ square of the distance between the centres of S_2 and S_3.

Hence, finally, if r_1, r_2, r_3 be the radii of the given circles S_1, S_2, S_3, and δ_1, δ_2, δ_3 the lengths of the sides of the triangle formed by joining their centres, we see that a circle which cuts S_1, S_2, S_3 at the respective angles θ_1, θ_2, θ_3 will always be touched by a variable circle $\lambda S_1+\mu S_2+\nu S_3$, provided λ, μ, ν satisfy the relation

$$\mathrm{A}\lambda^2+\mathrm{B}\mu^2+\mathrm{C}\nu^2+2\mathrm{F}\mu\nu+2\mathrm{G}\nu\lambda+2\mathrm{H}\nu\lambda=0,$$

where $$\mathrm{A}=r_1^2\sin^2\theta_1,\quad \mathrm{B}=r_2^2\sin^2\theta_2,\quad \mathrm{C}=r_3^2\sin^2\theta_3,$$
$$2\mathrm{F}=-(\delta_1^2-r_2^2-r_3^2+2r_2r_3\cos\theta_2\cos\theta_3),$$
$$2\mathrm{G}=-(\delta_2^2-r_3^2-r_1^2+2r_3r_1\cos\theta_3\cos\theta_1),$$
and $$2\mathrm{H}=-(\delta_3^2-r_1^2-r_2^2+2r_1r_2\cos\theta_1\cos\theta_2).$$

Again, if the angle of intersection of two circles be taken to be the definite angle subtended by the line joining the centres of the two circles at one of their common points of intersection, it is easily seen that the relation between λ, μ, ν, just obtained, is also precisely the

condition that $\lambda S_1 + \mu S_2 + \nu S_3$ shall touch V, a circle cutting S_1, S_2, S_3 *at the supplementary angles* $\pi-\theta_1, \pi-\theta_2, \pi-\theta_3$.

It follows, then, that two circles U and V which cut the given circles at angles $\theta_1, \theta_2, \theta_3$, and those supplemental, $\pi-\theta_1, \pi-\theta_2, \pi-\theta_3$ respectively, will *both be touched by a variable circle* $\lambda S_1 + \mu S_2 + \nu S_3$, when

$$\Sigma A\lambda^2 + 2\Sigma F\mu\nu = 0.$$

Conversely, by seeking the envelope of $\lambda S_1 + \mu S_2 + \nu S_3$ *under this condition, we find that the quartic*

$$\Sigma(BC - F^2)\Sigma^2 + 2\Sigma(GH - AF)S_2S_3 = 0,$$

where $S_1 = x^2+y^2+2g_1x+2f_1y+c_1$, $S_2 = \ldots\ldots$, $S_3 = \ldots\ldots$,

$$A = r_1^2 \sin^2\theta_1, \quad B = \ldots\ldots, \quad C = \ldots\ldots,$$

$$2F = -(\delta_1^2 - r_2^2 - r_3^2 + 2r_2r_3\cos\theta_2\cos\theta_3), \quad 2G = \ldots\ldots, \quad 2H = \ldots\ldots,$$

must represent a pair of circles U and V, *cutting* S_1, S_2, S_3 *at the given angles* $\theta_1, \theta_2, \theta_3$; $\pi-\theta_1, \pi-\theta_2, \pi-\theta_3$ *respectively.*

Again, if we change θ_1, for instance, into $\pi-\theta_1$, it is clear that the values of G and H will be altered; so that we shall get another quartic of the above form, giving the pair of circles which cut S_1, S_2, S_3 at the angles $\pi-\theta_1, \theta_2, \theta_3$ and $\theta_1, \pi-\theta_2, \pi-\theta_3$ respectively.

Hence, evidently, there will be in all *four* quartics of the above form; or, in other words, there are in general eight solutions to the problem, "To describe a circle to cut three given circles at given angles."

It may be observed that the radii of any one of these four pairs of circles may be obtained by writing down the relation (investigated by Prof. Cayley; see Dr. Salmon's "Conic Sections," p. 129, 5th ed.) which connects the mutual distances of any four points in a plane; and substituting therein $R^2 + r_1^2 - 2Rr_1\cos\theta_1$ for $(14)^2$, &c. The result is a quadratic in R; and similarly for the radii of any of the other pairs.

Remarks on the above Quartics.

Putting $\quad BC - F^2 = a$, &c., $\quad GH - AF = f$, &c.,

I propose to verify the fact that the quartic

$$\Omega = (a, b, c, 2f, 2g, 2h)(S_1, S_2, S_3)^2 = 0$$

breaks up into two circles.

It may be remarked, in the first place, that a curve of the form Ω is one of the fourth degree having a node at each of the two circular points at infinity, and that if it has two additional nodes, it will represent a pair of circles. (See a paper by Prof. Cayley, entitled "Investigations in Connexion with Carey's Equation," published in the "Quarterly Journal of Mathematics," No. 32, vol. viii.) Hence we have to show that the special quartic under consideration has two additional nodes.

1°. If the discriminant $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$ be supposed to vanish, Ω breaks up into linear factors, and so, obviously, represents a pair of circles.

2°. If the discriminant in question does not vanish, and Ω has two additional nodes, it is easy to show that they both lie on the Jacobian, J, of the given circles S_1, S_2, S_3; and in this case, therefore, we must prove that Ω meets J in two pairs of coincident points. Taking, then, the centre of the Jacobian as the origin of coordinates, we may write for the equations of the several circles J, S_1, S_2, S_3,

$$\left.\begin{aligned} J &= x^2+y^2-\rho^2 = 0 \\ S_1 &= x^2+y^2+2g_1x+2f_1y+\rho^2 = 0 \\ S_2 &= \ldots\ldots\ldots\ldots\ldots\ldots +\rho^2 = 0 \\ S_3 &= \ldots\ldots\ldots\ldots\ldots\ldots +\rho^2 = 0 \end{aligned}\right\} \text{ where } \rho = \text{radius of Jacobian,}$$

so that the points common to Ω and J (exclusive of the two circular points at infinity) coincide with the intersections of J and the conic whose equation is $(a, b, c, 2f, 2g, 2h)(x', y', z')^2 = 0$; where

$$\left.\begin{aligned} x' &= g_1x+f_1y+\rho^2 \\ y' &= g_2x+f_2y+\rho^2 \\ z' &= g_3x+f_3y+\rho^2 \end{aligned}\right\}.$$

Now, if we put $r_1^2 = A'$, $r_2^2 = B'$, $r_3^2 = C'$, $\delta_1^2 - r_2^2 - r_3^2 = -2F'$, &c., (where $\delta_1, \delta_2, \delta_3$; r_1, r_2, r_3 denote the same quantities as before,) it may be shown that the equation of J in terms of x', y', z' is

$$(a', b', c', 2f', 2g', 2h')(x', y', z')^2 = 0;$$

where $a' = B'C'-F'^2$, &c., as usual.

[This equation may be derived from the ordinary trilinear equation of a circle whose radius is r, and centre the point $(\alpha', \beta', \gamma')$; viz.,

$$\Sigma\{\beta'^2+\gamma'^2+2\beta'\gamma'\cos A-r^2\sin^2 A\}\alpha^2$$
$$+2\Sigma\{(\alpha'^2-r^2)\cos A-\beta'\gamma'-\gamma'\alpha'\cos C-\alpha'\beta'\cos B-r^2\cos B\cos C\}\beta\gamma = 0;$$

by writing $\frac{x'}{\delta'}$, $\frac{y'}{\delta''}$, $\frac{z'}{\delta'''}$ for α, β, γ, and expressing the several coefficients of x'^2, &c., in terms of given quantities.]

Hence we have to verify the fact that the conics given by the equations

$$(a', b', c', 2f', 2g', 2h')(x, y, z)^2 = 0,$$
$$(a, b, c, 2f, 2g, 2h)(x, y, z)^2 = 0,$$

have *double contact*; or, what is the same thing, that their reciprocals,

$$(A', B', C', 2F', 2G', 2H')(\lambda, \mu, \nu)^2 = 0,$$
$$(A, B, C, 2F, 2G, 2H)(\lambda, \mu, \nu)^2 = 0,$$

have double contact.

Now this is clearly the case; for we have

$$(A', B', C', 2F', 2G', 2H')(\lambda, \mu, \nu)^2 - (A, B, C, 2F, 2G, 2H)(\lambda, \mu, \nu)^2$$
$$= (r_1\cos\theta_1\lambda + r_2\cos\theta_2\mu + r_3\cos\theta_3\nu)^2.$$

Hence Ω meets J in a pair of two-fold points (exclusive of the circular points at infinity), or Ω has two additional nodes, and so breaks up into a pair of circles.*

The result just obtained enables us to find the radical axis of the pair of circles Ω; for this must evidently be the polar with respect to J, represented by $x^2+y^2-\rho^2=0$, of the point whose coordinates are given by the equations

$$\frac{g_1x+f_1y+\rho^2}{r_1\cos\theta_1}=\frac{g_2x+f_2y+\rho^2}{r_2\cos\theta_2}=\frac{g_3x+f_3y+\rho^2}{r_3\cos\theta_3}.$$

Hence we have the following theorem,—proved, I believe, by Plücker in his solution of the problem under consideration,—viz., that the

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* With reference to the above investigation, Prof. Cayley has remarked that "if $\Omega=(a,\ldots)(S_1,S_2,S_3)^2=0$ has a double point, then $J=J(S_1,S_2,S_3)=0$ passes through this double point;" but, he asks, "is it always true conversely, that if Ω, J have a twofold intersection, this is a double point of Ω?"

It may be replied that this is generally so, as I shall thus endeavour to show.

In the first place, there exists a quadric relation between

$$J=x^2+y^2-\rho^2,\ \text{and}\ S_1=x^2+y^2+2g_1x+2f_1y+\rho^2,\quad S_2=\ldots\ldots,\quad S_3=\ldots\ldots,$$

of the form $$(a',\ldots)(S_1,S_2,S_3)^2=kJ^2.$$

For if $g_1x+f_1y+\rho^2$, $g_2x+f_2y+\rho^2$, $g_3x+f_3y+\rho^2$ be written, for shortness, x',y',z', the quartic $(a',\ldots)(S_1,S_2,S_3)^2$ becomes

$$(a'+b'+c'+2f'+2g'+2h')\,J^2+4\left\{(a'+h'+g')\,x'+\ldots\ldots\right\}J+4\,(a',\ldots)(x',y',z')^2,$$

which will reduce to $(a'+\ldots)J^2$, provided a', b', ... satisfy certain linear equations. In fact, it will be found that when $(a',\ldots)(x',y',z')^2$ reduces to $(a'+b'+\ldots)\,\rho^2(\rho^2-x^2-y^2)$, the quartic function $(a',\ldots)(S_1,S_2,S_3)^2$ becomes at the same time $(a'+b'+\ldots)\times(x^2+y^2-\rho^2)^2$.

It follows from this, as in the analogous case of conics, that we can always determine θ so that the function $\Omega+\theta J^2$ breaks up into linear factors of S_1,S_2,S_3, *i. e.*, so that $$\Omega+\theta J^2=(\alpha S_1+\beta S_2+\gamma S_3)\,(\alpha'S_1+\beta'S_2+\gamma'S_3).$$

Hence, if J be supposed to have a twofold intersection with Ω, it is clear that θ may be so determined that one of the circles represented by the factors on the right-hand side of this equation, say $\alpha'S_1+\beta'S_2+\gamma'S_3$, shall *touch* J; but when a circle of the form $\alpha'S_1+\beta'S_2+\gamma'S_3$ touches J, it is easy to see that it reduces to a *point*.

If, therefore, the twofold intersection of Ω and J be taken as origin of coordinates, Ω becomes of the form

$$\Omega=(\alpha S_1+\beta S_2+\gamma S_3)\,(x^2+y^2)-\theta\,(x^2+y^2+2gx+2fy)^2;$$

where $x^2+y^2+2gx+2fy$ represents the Jacobian $J(S_1,S_2,S_3)$; or, in other words, a twofold intersection of Ω and J is a node of Ω.

Again, if θ can be so determined (as in the case of the quartic we have been considering) that $\Omega+\theta^2J^2$ is the square of a factor $\alpha S_1+\beta S_2+\gamma S_3$, it is evident that Ω then represents the two circles $\alpha S_1+\beta S_2+\gamma S_3+\theta J$, and $\alpha S_1+\beta S_2+\gamma S_3-\theta J$.

It may be observed that the nodes of Ω in the case before us coincide with the envelope-points $\lambda S_1+\mu S_2+\mu S_3=0$, where $\Sigma A'\lambda^2+2\Sigma F'\mu\nu=0$ and $\Sigma r_1\cos\theta_1\lambda=0$; *i. e.*, if $$S_1=(x-g_1)^2+(y-f_1)^2-r_1^2,\quad S_1=(x-g_2)^2+(y-f_2)^2-r_2^2,$$ and $$S_3=(x-g_3)^2+(y-f_3)^2-r_3^2,$$ the nodes of Ω are the points

$$x=\frac{\lambda g_1+\mu g_2+\nu g_3}{\lambda+\mu+\nu},\quad y=\frac{\lambda f_1+\mu f_2+\nu f_3}{\lambda+\mu+\nu};$$

where $\Sigma r_1^2\lambda^2-\Sigma\left\{(g_2-g_3)^2+(f_2-f_3)^2-r_2^2-r_3^2\right\}\mu\nu=0$, and $\Sigma r_1\cos\theta_1\lambda=0$.

radical axis of the pair of circles Ω coincides with an axis of similitude of three circles concentric respectively with S_1, S_2, S_3, and whose radii have to each other the ratios $r_1 \cos\theta_1 : r_2 \cos\theta_2 : r_3 \cos\theta_3$.

Again, by considering that a quartic of the form Ω is the envelope of a circle $\lambda S_1 + \mu S_2 + \nu S_3$ moving subject to a certain condition, we are readily led to another property of the four pairs of circles which cut three given circles at given angles ; viz.,

Any two of these pairs form four circles, which are all touched by each of four other circles orthogonal to the Jacobian J.

For instance, the two pairs of circles which cut S_1, S_2, S_3 at the angles (1) $\theta_1, \theta_2, \theta_3$; $\pi-\theta_1, \pi-\theta_2, \pi-\theta_3$; (2) $\pi-\theta_1, \theta_2, \theta_3$; $\theta_1, \pi-\theta_2, \pi-\theta_3$, are all touched by each of the circles $\lambda S_1 + \mu S_2 + \nu S_3$; where the ratios $\lambda : \mu : \nu$ are given by the two simultaneous equations

$$r_1^2 \sin^2\theta_1 \lambda^2 + r_2^2 \sin^2\theta_2 \mu^2 + r_3^2 \sin^2\theta_3 \nu^2 - (\delta_1^2 - r_2^2 - r_3^2 + 2r_2 r_3 \cos\theta_2 \cos\theta_3)\,\mu\nu$$
$$- (\delta_2^2 - r_3^2 - r_1^2 + 2r_3 r_1 \cos\theta_3 \cos\theta_1)\,\nu\lambda$$
$$- (\delta_3^2 - r_1^2 - r_2^2 + 2r_1 r_2 \cos\theta_1 \cos\theta_2)\,\lambda\mu = 0,$$

$$r_1^2 \sin^2\theta_1 \lambda^2 + r_2^2 \sin^2\theta_2 \mu^2 + r_3^2 \sin^2\theta_3 \nu^2 - (\delta_1^2 - r_2^2 - r_3^2 + 2r_2 r_3 \cos\theta_2 \cos\theta_3)\,\mu\nu$$
$$- (\delta_2^2 - r_3^2 - r_1^2 - 2r_3 r_1 \cos\theta_3 \cos\theta_1)\,\nu\lambda$$
$$- (\delta_3^2 - r_1^2 - r_2^2 - 2r_1 r_2 \cos\theta_1 \cos\theta_2)\,\lambda\mu = 0.$$

The bicircular quartic $\Omega = (a, b, c, 2f, 2g, 2h)(S_1, S_2, S_3)^2 = 0$ has hitherto been regarded as an *envelope* ; *i.e.*, the envelope of a variable circle $\lambda S_1 + \mu S_2 + \nu S_3$, where λ, μ, ν satisfy a certain relation.

Now it is evident that it may be also looked upon as a *locus* ; viz., the locus of a pair of points $\frac{S_1}{\alpha} = \frac{S_2}{\beta} = \frac{S_3}{\gamma}$;* where α, β, γ are subject to the relation $(a, b, c, 2f, 2g, 2h)(\alpha, \beta, \gamma)^2 = 0$.

Hence we have the following theorem :—The two circles represented by the quartic $\Omega = (a, b, c, 2f, 2g, 2h)(S_1, S_2, S_3)^2 = 0$ are the locus of a pair of points inverse to each other with respect to the Jacobian J ; viz., the points $\frac{S_1}{\alpha} = \frac{S_2}{\beta} = \frac{S_3}{\gamma}$; where α, β, γ satisfy the equation $(a, b, c, 2f, 2g, 2h)(\alpha, \beta, \gamma)^2 = 0$; or, in other words, they are circles inverse to each other with respect to J.

We may briefly notice two special cases of the quartic Ω ; viz., α° when S_1, S_2, S_3 meet in a point ; β° when S_1, S_2, S_3 are coaxal.

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* *Note by* Prof. Cayley.—"This is an interesting kind of geometrical correspondence, deserving to be studied ; viz., to a point (α, β, γ) there correspond the *pair* of points $\frac{S_1}{\alpha} = \frac{S_2}{\beta} = \frac{S_3}{\gamma}$ (where $S_1 = 0$, $S_2 = 0$, $S_3 = 0$ are arbitrary circles) ; and, therefore, in general, to the curve $(\alpha, \beta, \gamma)^n = 0$ a curve $(S_1, S_2, S_3)^n = 0$ of the order $2n$."

$\alpha°$. In this case $\rho = 0$; and the Jacobian reduces to a point, through which the envelope-circle of Ω always passes. Hence it is clear that Ω breaks up into two factors of the forms x^2+y^2 and $x^2+y^2+2gx+2fy+c$. Under these circumstances, therefore, the number of solutions reduces to *four*.

$\beta°$. If S_1, S_2, S_3 meet in two common points, a quartic of the form $\Omega=0$ represents a pair of circles, both passing through each of these points,—whatever the values of the coefficients A, B, C, &c.—but in this case there is a certain relation between the cosines of the angles θ, ϕ, ψ at which any circle cuts S_1, S_2, S_3 (see Dr. Salmon's "Conic Sections," p. 103, Ex. 9, 5th edit.); and unless this is satisfied by the given angles $\theta_1, \theta_2, \theta_3$, the problem is not a possible one. If we were to make use of this relation, and express the quartic Ω in terms of the coordinates x and y, we should find that Ω vanishes identically; the result is, therefore, indeterminate.

Particular cases.—1. If $\theta_1 = 0$ or π, $\theta_2 = 0$ or π, $\theta_3 = 0$ or π, we have Casey's Equations for the four pairs of circles *touching* three given circles. (See the paper by Prof. Cayley, referred to above.)

2. If $\theta_1 = \theta_2 = \theta_3 = \frac{1}{2}\pi$, we shall get the unique equation

$$\mathrm{KJ}^2 = \Sigma\{(\delta_1^2 - r_2^2 - r_3^2)^2 - 4r_2^2 r_3^2\}\, S_1^2 - 2\,.\,\Sigma\{(\delta_2^2 - r_3^2 - r_1^2)(\delta_3^2 - r_1^2 - r_2^2) + 2r_1^2(\delta_1^2 - r_2^2 - r_3^2)\}\, S_2 S_3 = 0.$$

3. If $\alpha_1, \alpha_2, \alpha_3$ be the angles of intersection (*i.e.*, the definite angles between the radii at a common point) of the pairs of circles S_2, S_3; S_3, S_1; S_1, S_2; and $\alpha_1+\alpha_2+\alpha_3 = 2p$; the equations of the four pairs of circles which pass through the intersections of S_1, S_2, S_3 are

$$\Sigma\,.\,r_1 \sin(p-\alpha_1)\,.\,S_2S_3 = 0,$$

$$-r_1 \sin p\,.\,S_2S_3 + r_2 \sin(p-\alpha_3)\,.\,S_3S_1 + r_3 \sin(p-\alpha_2)\,.\,S_1S_2 = 0,$$

$$r_1 \sin(p-\alpha_3)\,.\,S_3S_1 - r_2 \sin p\,.\,S_1S_2 + r_3 \sin(p-\alpha_1)\,.\,S_1S_2 = 0,$$

$$r_1 \sin(p-\alpha_2)\,.\,S_3S_1 + r_2 \sin(p-\alpha_1)\,.\,S_1S_2 - r_3 \sin p\,.\,S_1S_2 = 0;$$

where, as before, r_1, r_2, r_3 are the radii of S_1, S_2, S_3.

These results may be obtained by putting the coefficients of S_1^2, S_2^2, S_3^2 in the quartic Ω, each equal to zero; and so eliminating $\theta_1, \theta_2, \theta_3$.

I may conclude with the remark that the above method may be employed with advantage to find the equations of the spheres which can be drawn to cut four given spheres at given angles $\theta_1, \theta_2, \theta_3, \theta_4$. For we may show that a pair of spheres cutting four spheres given by the equations

$$S_1 = x^2+y^2+z^2+2g_1x+\&c. = 0,$$
$$S_2 =$$
$$\&c.,$$

at the angles $\theta_1, \theta_2, \theta_3, \theta_4$, and their supplements respectively (measured in the definite way indicated before) will be always touched by a variable sphere $\alpha S_1+\beta S_2+\gamma S_3+\delta S_4$,

where $\alpha, \beta, \gamma, \delta$ satisfy the relation

$$\Sigma A\alpha^2+2L\beta\gamma+2M\gamma\alpha+2N\alpha\beta+2P\alpha\delta+2Q\beta\delta+2R\gamma\delta = 0;$$

and, therefore, conversely, we have to find the envelope of the variable sphere in question when $\alpha, \beta, \gamma, \delta$ are subject to the relation just given.

The result is, that there are in all eight pairs of spheres satisfying the given conditions, and the radii of any pair may be found by writing down the relation which connects the mutual distances of five points in space.

APPENDIX.

1. Since writing the above, I have succeeded in arriving at the following equations for the groups of circles cutting three given small circles S_1, S_2, S_3 on a sphere at given angles $\theta_1, \theta_2, \theta_3$; viz.,

If the equations of the given circles be written in the forms

$$S_1 = \sqrt{S-a_1}\sec r_1,$$
$$S_2 = \sqrt{S-a_2}\sec r_2,$$
$$S_3 = \sqrt{S-a_3}\sec r_3,$$

(see Dr. Salmon's "Geometry of Three Dimensions," p. 195); and

$$\tan^2 r_1 \sin^2\theta_1 = A, \quad \tan^2 r_2 \sin^2\theta_2 = B, \quad \tan^2 r_3 \sin^2\theta_3 = C,$$
$$-2(-\cos\delta_1+\cos r_2\cos r_3+\sin r_2\sin r_3\cos\theta_2\cos\theta_3)\sec r_2\sec r_3 = 2F,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots = 2G,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots = 2H;$$

where r_1, r_2, r_3 are the radii of the given circles, and $\delta_1, \delta_2, \delta_3$ the distances between their centres; then the equation

$$\begin{vmatrix} A, & H, & G, & S_1 \\ H, & B, & F, & S_2 \\ G, & F, & C, & S_3 \\ S_1, & S_2, & S_3, & 0 \end{vmatrix} = 0$$

belongs to a group of circles cutting S_1, S_2, S_3 at the angles $\theta_1, \theta_2, \theta_3$ respectively.

This result is obtained in the same manner as in the case of circles on a plane. In the first place, if the angle at which two circles cut is measured by the angle which the great circle joining their centres subtends at the point of meeting, and U denote the circle cutting S_1, S_2, S_3 at the angles $\theta_1, \theta_2, \theta_3$ (measured in this way), we may show that U is always touched by a variable circle $\lambda S_1+\mu S_2+\nu S_3$, where λ, μ, ν satisfy a quadric relation of the form

$$A\lambda^2+B\mu^2+C\nu^2+2F\mu\nu+2G\nu\lambda+2H\lambda\mu = 0;$$

where A, B, C, 2F, 2G, 2H are invariant functions of the radii and distances between the centres of the given circles.

To simplify our investigation as much as possible, let us take for the equations of U, S_1, S_2, S_3 the following; viz.,

$$\begin{aligned} U &= \sqrt{S} - z \sec R = 0, \\ S_1 &= \sqrt{S} - (l'x + m'y + n'z) \sec r_1 = 0, \\ S_2 &= \sqrt{S} - (l''x + m''y + n''z) \sec r_2 = 0, \\ S_3 &= \sqrt{S} - (l'''x + m'''y + n'''z) \sec r_3 = 0; \end{aligned}$$

where

$$\begin{aligned} l'l''' + m''m''' + n''n''' &= \cos \delta_1, \\ . \quad . \quad . \quad &= \cos \delta_2, \\ . \quad . \quad . \quad &= \cos \delta_3; \end{aligned}$$

and, according to the conditions of the problem,

$$\begin{aligned} n' &= \cos R \cos r_1 + \sin R \sin r_1 \cos \theta_1, \\ n'' &= \cos R \cos r_2 + \sin R \sin r_2 \cos \theta_2, \\ n''' &= \cos R \cos r_3 + \sin R \sin r_3 \cos \theta_3. \end{aligned}$$

If, therefore, we write for $\lambda S_1 + \mu S_2 + \nu S_3 = 0$ the equation

$$(\lambda + \mu + \nu) \sqrt{S} - \Sigma \lambda l' \sec r_1 . x - \Sigma \lambda m' \sec r_1 . y - \Sigma \lambda n' \sec r_1 . z = 0;$$

where $\Sigma \lambda l' \sec r_1 = \lambda l' \sec r_1 + \mu l'' \sec r_2 + \nu l''' \sec r_3$, &c.;

we see that

$$\begin{aligned} \Sigma \lambda l' \sec r_1 . x + \Sigma \lambda m' \sec r_1 . y \\ + \{ \lambda (n' \sec r_1 - \sec R) + \mu (n'' \sec r_2 - \sec R) \\ + \nu (n''' \sec r_3 - \sec R) \} . z = 0; \end{aligned}$$

or, say, $Lx + My + Nz = 0$ represents the plane through the origin and the two points common to U and $\lambda S_1 + \mu S_2 + \nu S_3$. Hence the condition that these circles shall *touch* becomes

$$\frac{N}{\sqrt{L^2 + M^2 + N^2}} = \sin R, \quad \text{or} \quad N^2 \cos^2 R = (L^2 + M^2) \sin^2 R;$$

which, ultimately, since

$$\begin{aligned} n' \sec r_1 &= \cos R + \sin R \tan r_1 \cos \theta_1, \\ n'' \sec r_2 &= \quad . \quad . \quad . \quad . \quad , \\ n''' \sec r_3 &= \quad . \quad . \quad . \quad . \quad , \end{aligned}$$

reduces to $A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0$,

where $A = \tan^2 r_1 \sin^2 \theta_1$, $B = \tan^2 r_2 \sin^2 \theta_2$, $C = \tan^2 r_3 \sin^2 \theta_3$, and

$$\begin{aligned} 2F &= -2 (-\cos \delta_1 + \cos r_2 \cos r_3 + \sin r_2 \sin r_3 \cos \theta_2 \cos \theta_3) \sec r_2 \sec r_3, \\ 2G &= \quad . \quad . \quad . \quad . \quad . \quad , \\ 2H &= \quad . \quad . \quad . \quad . \quad . \quad . \end{aligned}$$

Again, if we take a circle V, which cuts S_1, S_2, S_3 at the supplementary angles $\pi - \theta_1$, $\pi - \theta_2$, $\pi - \theta_3$ (measured in the way above explained), it may be proved in the same manner that V is always

touched by the circle $\lambda S_1+\mu S_2+\nu S_3$, where λ, μ, ν satisfy the quadric relation just obtained.

Conversely, if

$$S_1=\sqrt{S-a_1}\sec r_1, \quad S_2=\sqrt{S-a_2}\sec r_2, \quad S_3=\sqrt{S-a_3}\sec r_3,$$

the envelope $\Sigma\,(BC-F^2)\,S_1^2+2\Sigma\,(GH-AF)\,S_2S_3=0$

gives a pair of circles cutting S_1, S_2, S_3 at the angles $\theta_1, \theta_2, \theta_3$ and their supplements $\pi-\theta_1$, $\pi-\theta_2$, $\pi-\theta_3$ respectively.

Particular case.—If $\theta_1=0$, $\theta_2=0$, and $\theta_3=0$, for instance, we have Casey's Equation (see Dr. Salmon's "Conic Sections," p. 345) for a pair of circles *touching* S_1, S_2, S_3, either all externally or all internally; viz.,

$$\Sigma\,.\,\sqrt{\{-\cos\delta_1+\cos(r_2-r_3)\}\sec r_2\sec r_3}\,.\,\{\sqrt{S-a_1}\sec r_1\}=0.$$

2. If

$$\begin{aligned} S_1&=\sqrt{S-(l_1x+m_1y+n_1z)}=0,\\ S_2&=\sqrt{S-(l_2x+m_2y+n_2z)}=0,\\ S_3&=\sqrt{S-(l_3x+m_3y+n_3z)}=0, \end{aligned}$$

represent three given conics, each having double contact with a given conic $S=x^2+y^2+z^2=0$,

the above results point to the more general equation

$$\begin{vmatrix} A, & H, & G, & S_1\\ H, & B, & F, & S_2\\ G, & F, & C, & S_3\\ S_1, & S_2, & S_3, & 0 \end{vmatrix}=0,$$

where $A=(l_1^2+m_1^2+n_1^2-1)\sin^2\theta_1$, $B=\ldots\ldots$, $C=\ldots\ldots$,

$$\begin{aligned} 2F=-2\{&1-(l_2l_3+m_2m_3+n_2n_3)\\ &+\sqrt{(l_2^2+m_2^2+n_2^2-1)(l_3^2+m_3^2+n_3^2-1)}\,.\cos\theta_2\cos\theta_3\}, \end{aligned}$$

$2G=\ldots\ldots$, $2H=\ldots\ldots$

This belongs to a group of conics $S-(lx+my+nz)^2$, whose invariants are connected with those of the given conics S_1, S_2, S_3 by the equations

$$\frac{ll_1+mm_1+nn_1-1}{\sqrt{(l^2+m^2+n^2-1)(l_1^2+m_1^2+n_1^2-1)}}=\cos\theta_1, \quad \ldots=\cos\theta_2, \quad \ldots=\cos\theta_3.$$

I am not sure, however, that I can give the geometric meaning of these relations.

The President next requested assistance in the solution of a "Question in the Mathematical Theory of Vibrating Strings," which he had been unable to solve.

A string is said to execute a forced vibration when it is compelled to perform vibrations synchronous with those of a vibrating body to which one end is attached. The amplitude of the forced vibrations is

greatest when the length or tension of the string is so adjusted that its natural period of vibration, for its fundamental note, or one of its harmonics, is the same as that of the forcing body. And, within limits, the amplitude diminishes as the period of the string diverges from that of the body.

The theory of forced vibrations, when the motion of the forcing body is transverse to the length of the string, has been fully discussed by various writers, and especially by Helmholz and Dordini. The latter of these has shown how it is that the amplitude of vibration of the string may vary while that of the body remains constant; and, further, he has given expressions which do not lead to the absurd conclusion (resulting from the ordinary formulæ) of an infinite amplitude when the periods of string and body are absolutely synchronous.

The expressions in question, or others easily deducible from them, when applied to the discussion of the nodes, and length of vibrating segments, explain the phenomena which may be observed in experiment, of nodes of least motion rather than of perfect rest, and of segments of varying length.

But of the forced vibrations, when the forcing body moves in the direction of the string's length, there is, so far as the speaker was aware, as yet no mathematical theory. The principal phenomenon is well known, viz., that when the length and tension of the string are such that it gives out the same note as the forcing body if the forcing motion is transverse, it will give out the octave below when the forcing motion is longitudinal.

In this problem there arise two *primâ facie* difficulties; first, that a forcing motion may be conceived, and, indeed, may be experimentally set up, without producing a vibration in the string; and, secondly, that a mechanical vibration of a given period can give rise to a vibration of a period double of the former. The key to the explanation of these difficulties is probably to be found in the consideration, corroborated by experience, that the motion of the string depends upon small quantities of the second order.

A mathematical solution of the question is the desideratum suggested.

Mr. Strutt made a few remarks on the subject, and mentioned some results he had arrived at. A communication from Prof. Cayley, respecting the extension of the Society's sphere of action, was laid before the meeting by the President; it was determined that the matter should be discussed at the next meeting of the Society. Prof. Clerk-Maxwell asked for information from the members as to the convention established among Mathematicians, with respect to the relation between the positive direction of motion along any axis, and the positive direction of rotation round it. In Sir W. R. Hamilton's Lectures on Quaternions, the coordinate axes are drawn x to South, y to West, and z upwards.

The same system is adopted in Prof. Tait's Quaternions, and in Listing's "Vorstudien zur Topologie." The positive directions of translation and of rotation are thus connected in a left-handed screw, or the tendril of the hop.

On the other hand, in Thomson's and Tait's Natural Philosophy, § 234, the relations are defined with reference to a watch, and lead to the opposite system, symbolized by an ordinary or right-handed screw, or the tendril of the vine. If the actual rotation of the earth from West to East be taken positive, the direction of the earth's axis from South to North is positive in this system. In pure mathematics little inconvenience is felt from this want of uniformity; but in astronomy, electro-magnetics, and all physical sciences, it is of the greatest importance that one or other system should be specified and persevered in. The relation between the one system and the other is the same as that between an object and its reflected image, and the operation of passing from the one to the other has been called by Listing *Perversion*.

Sir W. Thomson and Dr. Hirst stated the arguments in favour of the right-handed system, derived from the motion of the earth and planets and the convention that North is to be reckoned positive, and also from the practice of Mathematicians, in drawing x to the right-hand and y upwards on the plane of the black board, and z towards the spectator. No arguments in favour of the opposite system being given, the right-handed system, symbolized by a corkscrew or the tendril of the vine, was adopted by the Society.

The following presents were received:—

"Crelle's Journal," 73 Band, zweites Heft, April, 1871.

"Proceedings of the Royal Society," vol. xix. No. 127.

"Monatsbericht," Feb., März, 1871.

"Journal of London Institution," No. 5.

"Journal of Institute of Actuaries," No. lxxxii., January, 1871.

"Jahrbuch über die gesammten Fortschritte der Mathematik," erster Band, 1868: from Dr. Karl Ohrtmann and Dr. Felix Müller.

"Theorie und Anwendung des sogenannten Variationscalcus, von Dr. G. W. Strauch," (1849): from Mr. C. R. Hodgson, B.A.

June 8th, 1871.

W. SPOTTISWOODE, Esq., F.R.S., President, in the Chair.

Visitors, Messrs. C. Smith, M.A., J. W. L. Glaisher, B.A., and V. Dwelshauvers-Dery, Professeur de Mécanique appliquée à Liège.

Messrs. W. Chadwick, M.A., Fellow of Corpus Christi College, and J. Griffiths, M.A., Fellow of Jesus College, Oxford, were elected Members of the Society.

Prof. Cayley gave an account of Plücker's Models of certain Quartic Surfaces.

At the Meeting of the British Association at Nottingham, in 1866, Prof. Plücker read his paper on "Complexes of the Second Order" (for abstract, see Report of British Association, 1866). On this occasion he showed a series of Models, constructed by Epkens, of Bonn, to illustrate his theory. Prof. Plücker had copies of these models made for Dr. Hirst, who presented them to the Society, at the December meeting in 1866.

On Plücker's Models of certain Quartic Surfaces.

By Prof. Cayley, F.R.S., V.P.

The Society possesses a series of 14 wooden models of surfaces, constructed under the direction of the late Prof. Plücker, in illustration of the theory developed in his posthumous work, "Neue Geometrie des Raumes gegründet auf die Betrachtung der geraden Linie als Raumelemente," Leipzig, 1869. These all of them represent, I believe, Equatorial Surfaces; viz., models 1 to 8 represent cases of the 78 forms of equatorial surfaces "deren Breiten-Curven eine feste Axenrichtung besitzen," vol. ii. pp. 352—363, the remaining models, Nos. 9 to 14, I have not completely identified. I propose to go into the theory only so far as is required for the explanation of the models.

In a "Complex," or triply infinite system of lines, there is in any plane whatever a singly infinite system of lines enveloping a curve; and if we attend only to the curves the planes of which pass through a given fixed line, the locus of these curves is a "complex surface." Similarly, there is through any point whatever a single infinite series of lines generating a cone; and if we attend only to the cones which have their vertices in the given fixed line, then the envelope of these cones is the same complex surface. In the case considered of a complex of the second degree, the curves and cones are each of them of the second order; the fixed line is a double line on the surface, so that (attending to the first mode of generation) the complete section by any plane through the fixed line is made up of this line twice, and of a conic; the surface is thus of the order 4: it is also of the class 4; the surface has, in fact, the nodal line, and also 8 nodes (conical points), and we have thus a reduction = 32 in the class of the surface.

In the particular case where the nodal line is at infinity, the complex surface becomes an equatorial surface; viz., (attending to the first mode

of generation) we have here a series of parallel planes each containing a conic, and the locus of these conics is the equatorial surface.

It is convenient to remark that, taking a, b, h to be homogeneous functions of (x, w) of the order 2; f, g of the order 1; and c of the order 0 (a constant); then the equation of a complex surface (see vol. i. p. 162) is

$$\begin{vmatrix} & y & z & 1 \\ y & a & h & g \\ z & h & b & f \\ 1 & g & f & c \end{vmatrix} = 0;$$

and that, writing $w=1$, or considering a, h, b; f, g; c as functions of x of the orders 2, 1, 0 respectively, we have an equatorial surface.

A particular form of equatorial surface is thus $bcy^2+caz^2+ab=0$, or taking $c=1$, this is $by^2+az^2+ab=0$, where a, b are quadric functions of x.

The surface is still, in general, of the fourth order: it may however degenerate into a cubic surface, or even into a quadric surface; the last case is however excluded from the enumeration. The section by any plane parallel to that of yz is a conic; the section by the plane $y=0$ is made up of the pair of lines $a=0$, and of the conic $z^2+b=0$; that by the plane $z=0$ is made up of the pair of lines $b=0$, and of the conic $y^2+a=0$; the last mentioned planes may be called the principal planes, and the conics contained in them principal conics. The surface is thus the locus of a variable conic, the plane of which is parallel to that of yz, and which has for its vertices the intersections of its plane with the two principal conics respectively. And we have thus the particular equatorial surfaces considered by Plücker, vol ii. pp. 346—363 (as already mentioned), under the form

$$\frac{y^2}{Ex^2+2Ux+C}+\frac{z^2}{Fx^2-2Rx+B}+1=0,$$

and of which he enumerates 78 kinds; viz., these are—

1	to	17	Principal conics each proper.
18	„	29	One of them a line-pair.
30	„	32	Each a line-pair.
33	„	39	Principal conics, each proper, but having a common point.
40	„	43	One of them a line-pair, its centre on the other principal conic.
44	„	61	One principal conic a parabola.
62	„	73	One principal conic a pair of parallel lines.
74	„	76	Principal conics each a parabola.
77 and 78			Principal conics, one of them a parabola, the other a pair of parallel lines.

The models 1 to 8 correspond hereto, as follows :—

Mod.	1	is 13,	*i. e.*	2 is Mod.	5
„	2	9		3 „	6
„	3	40		4 „	7
„	4	34		9 „	2
„	5	2		13 „	1
„	6	3		32 „	8
„	7	4		34 „	4
„	8	32		40 „	3.

Mod. 5 is 2 : the form of the equation is here

$$\frac{y^2}{l^2[(x-\alpha)^2+\beta^2]} - \frac{z^2}{l'^2[(x-\alpha')^2+\beta'^2]} = 1;$$

viz., the principal conics are one of them a hyperbola, the other imaginary ; hence the generating conic has always two, and only two, real vertices, viz., it is always a hyperbola : there are no real lines.

Mod. 6 is 3 : the form of the equation is

$$\frac{y^2}{l^2[(x-\alpha)^2+\beta^2]} + \frac{z^2}{l'^2[(x-\alpha')^2+\beta'^2]} = 1;$$

viz., the principal conics are each of them a hyperbola ; the generating conic has four real vertices, viz., it is always an ellipse : there are no real lines.

Mod. 7 is 4 : the form of the equation is

$$\frac{y^2}{l^2(x-\gamma)(x-\delta)} + \frac{z^2}{l'^2[(x-\alpha')^2+\beta'^2]} + 1 = 0.$$

The principal conics are one of them an ellipse, the other imaginary ; for values of x between γ and δ, the variable conic has two real vertices or it is a hyperbola ; for any other values it is imaginary, so that the surface lies wholly between the planes $x=\gamma$, $x=\delta$: the surface contains the real lines $y=0$, $x=\gamma$ and $y=0$, $x=\delta$.

Mod. 2 is 9 : the form of the equation is

$$\frac{y^2}{l^2(x-\gamma)(x-\delta)} + \frac{z^2}{l'^2(x-\gamma')(x-\delta')} + 1 = 0,$$

where, say the values γ, δ lie between the values γ', δ' : the principal conics are each of them an ellipse, the vertices (on the axis or line $y=0$, $z=0$) of the one ellipse lying between those of the other ellipse. The variable conic for values of x between γ, δ has four real vertices, or it is an ellipse ; for values beyond these limits, but within the limits γ', δ'—say from γ to γ', and from δ to δ'—there are two real vertices, or the conic is a hyperbola ; and for values beyond the limits γ', δ', the variable conic is imaginary.

There are four real lines ($y=0$, $x=\gamma$), ($y=0$, $x=\delta$), ($z=0$, $x=\gamma'$), ($z=0$, $x=\delta'$). The surface consists of a central pillow-like portion,

joined on by two conical points to an upper portion, and by two conical points to an under portion, the whole being included between the planes $x=\gamma'$, $x=\delta'$.

Mod. 1 is 13: the form of the equation is

$$\frac{y^2}{l^2(x-\gamma)(x-\delta)} - \frac{z^2}{l'^2(x-\gamma')(x-\delta')} + 1 = 0;$$

the values γ', δ' lying between γ, δ; the principal conics are one of them a hyperbola, the other an ellipse, the vertices (on the axis or line $y=0$, $z=0$) of the hyperbola lying between those of the ellipse.

The variable conic, for values of x between γ', δ', has two real vertices, or it is a hyperbola; for the values, say, from γ' to γ, and δ' to δ, there are four real vertices, or the conic is an ellipse; for values beyond the limits γ, δ, there are two real vertices, and the conic is a hyperbola. There are the four real lines $(y=0, x=\gamma)$, $(y=0, x=\delta)$, and $(z=0, x=\gamma')$, $(z=0, x=\delta')$. The surface consists of 8 portions joined to each other by 8 conical points, but the form can scarcely be explained by a description.

Mod. 8 is 32: the form of the equation is,

$$\frac{y^2}{l^2(x-\gamma)^2} + \frac{z^2}{l'^2(x-\gamma')^2} = 1;$$

viz., the principal conics are each of them a line-pair, the variable conic is always an ellipse.

There are the two real nodal lines $(y=0, x=\gamma)$ and $(z=0, x=\gamma')$, each of these being in the neighbourhood of the axis crunodal, and beyond certain limits acnodal; the surface is a scroll, being, in fact, the well known surface which is the boundary of a small circular pencil of rays obliquely reflected, and consequently passing through two focal lines.

Mod. 4 is 34: the equation is,

$$\frac{y^2}{l^2(x-\gamma)(x-\delta)} + \frac{z^2}{l'^2(x-\gamma')(x-\delta)} + 1 = 0,$$

where $x=\delta$ is not intermediate between the values $x=\gamma$ and $x=\gamma'$; say the order is δ, γ, γ'. The surface is thus a *cubic* surface; the principal conics are ellipses having on the axis a common vertex at the point $x=\delta$, and the remaining two vertices on the same side of the last-mentioned one. The variable conic for values between δ and γ has four real vertices, or it is an ellipse; for values between γ and γ' two real vertices, or it is a hyperbola; and for values beyond the limits δ, γ' it is imaginary. There are on the surface the two real lines $(y=0, x=\gamma)$ and $(z=0, x=\gamma')$. The surface consists of a finite portion joined on by two conical points to the remaining portion.

Mod. 3 is 40: the form of equation is

$$\frac{y^2}{l^2(x-\gamma)(x-\delta)} + \frac{z^2}{l'^2(x-\delta)^2} + 1 = 0.$$

The surface is thus a *cubic* surface; the principal conics are, one of them an ellipse, the other a pair of imaginary lines intersecting on the ellipse; for values of x between γ and δ, the variable conic has thus two real vertices, and it is a hyperbola; for values beyond these limits it is imaginary, and the whole surface is thus included between the planes $x=\gamma$ and $x=\delta$. There are the two real lines ($y=0$, $x=\gamma$) and ($z=0$, $x=\delta$).

Taking $l^2=l'^2=1$, the surface is,

$$\frac{y^2}{(x-\gamma)(x-\delta)} + \frac{z^2}{(x-\delta)^2} + 1 = 0,$$

which is a *particular* case of the parabolic cyclide.

As already mentioned, I have not completely identified the remaining models 9 to 14, but I will say a few words about them.

The equatorial surfaces, not included in the preceding 78 cases, Plücker distinguishes (vol. ii. p. 363) as "gedrehte" or "tordirte," say as twisted equatorial surfaces; the equation of such a surface is

$$by^2+2hyz+az^2+ab-h^2=0,$$

where
$$b = Fx^2-2Rx+B,$$
$$a = Ex^2+2Ux+C,$$
$$h = Kx^2-Ox-G \text{ (or in particular } =-Ox-G).$$

Mod. 13 is such a surface, being a twisted form of model 2.

Mod. 9 and Mod. 14 belong, I think, to the case $a=0$; viz., the form of the equation is $by^2+2hyz-h^2=0$. The variable conic is a hyperbola, the direction of one of the asymptotes being constant (vol. ii. p. 368).

There are moreover (p. 372) equatorial surfaces in which the variable conic is always a parabola, and where there are on the surface four real or imaginary lines.

Mods. 10, 11, and 12 seem to represent such surfaces.

For future use, it was agreed, in consequence of the author's suggestion, that tickets (in addition to those at present on the models) should be attached, in accordance with the identifications in Prof. Cayley's communication.

An account of the following paper was then given by the Author:—

On the Motion of a Plane under Certain Conditions.

By S. Roberts, M.A.

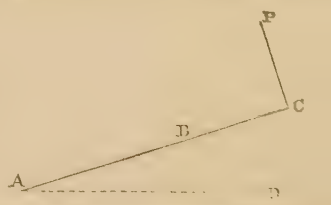
Point directrices

1. Imagine an indefinite plane inclined at an invariable angle to a fixed plane, in which two of the points of the former plane describe given curves. Every point of the moving plane will describe a curve lying in a plane parallel to the fixed plane, and the general trajectory will be the locus of a point rigidly connected with a line of constant length moving with its extremities on the given curves as directrices. With reference to space of three dimensions, the motion constitutes a very restricted case, and the moving plane envelopes a developable.* But if the angle of inclination is evanescent, the motion is that of a plane in plane space, determined by the motions of two points of the plane, and is therefore general.

The primary elements of a plane are points and lines. And if the directrices above mentioned are algebraical, problems of the following kind arise. Given the degree, class, and general singularities of the directrices, to determine the degree, class, and general singularities, (1) of the trajectory of any point of the moving plane, (2) of the envelope of any right line in the plane. Other questions occur as to the effect of the directrices becoming identical, and of their passing through or being otherwise specially related to the circular points at infinity, or to the line at infinity. It is also desirable to investigate the loci of the instantaneous centres relative to the moving and fixed planes; since the motion may also be generated by the rolling of the former curve on the latter. The subject is evidently extensive, and I am able to give little more than the commencement of a theory.

2. With reference to the trajectory of a point, the type of curve to be considered is the locus of a point rigidly connected with a line of constant length, one extremity of which moves on a curve C_m of the degree m, and the other on a curve $C_{m'}$ of the degree m', the whole of the conditions relating to plane space.

Write $\phi(x, y) = 0$ for the equation of C_m, $\psi(x, y) = 0$ for that of $C_{m'}$. Let x_1, y_1 be the coordinates of the point A on C_m; x_2, y_2 those of the point B on $C_{m'}$; x, y those of P, the describing point. Also let PC (perpendicular to AB at C) $= c$, AC $= a$, BC $= b$, $\angle$BAD (the angle which AB makes with the axis of x) $= \theta$.



* The chief characteristics of this developable can be determined by means of the results hereafter obtained.

Then, the axes being rectilinear, we have

$$x_1 = x - a\cos\theta + c\sin\theta,$$
$$y_1 = y - a\sin\theta - c\cos\theta,$$
$$x_2 = x - b\cos\theta + c\sin\theta,$$
$$y_2 = y - b\sin\theta - c\cos\theta.$$

Consequently, by the conditions

$$\phi(x_1, y_1) = 0, \quad \psi(x_2, y_2) = 0,$$

the problem is reduced to the elimination of θ from

$$\left.\begin{aligned} \phi(x - a\cos\theta + c\sin\theta, \quad y - a\sin\theta - c\cos\theta) &= 0 \\ \psi(x - b\cos\theta + c\sin\theta, \quad y - b\sin\theta - c\cos\theta) &= 0 \end{aligned}\right\} \ldots\ldots (1).$$

In fact, these two equations may be taken to represent the curve.

We may also write the equivalent system,

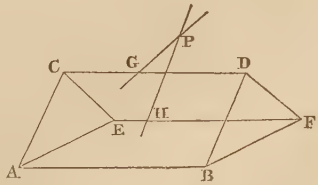
$$\left.\begin{aligned} \phi(x - aX + cY, \quad y - aY - cX) &= 0 \\ \psi(x - bX + cY, \quad y - bY - cX) &= 0 \\ X^2 + Y^2 &= 1 \end{aligned}\right\} \ldots\ldots\ldots\ldots (2).$$

3. It is worth while to note an immediate consequence of the forms of the equations (1).

The quantities $a\cos\theta - c\sin\theta$, $a\sin\theta + c\cos\theta$ may be taken as coordinates of a point of the circle $x^2 + y^2 = a^2 + c^2$, and in like manner the quantities $b\cos\theta - c\sin\theta$, $b\sin\theta + c\cos\theta$ may be taken as coordinates of a point of the circle $x^2 + y^2 = b^2 + c^2$. The angle between the radii drawn from the common centre of the circles to two corresponding points (corresponding, that is to say, to the same value of θ) remains constant, while θ varies. In fact, the angle is equal to the angle subtended by the moving line AB at P the describing point.

Hence, we have the following result :—If, in a given position of the radii, the curve C_m be translated so that the point (in its plane and carried with it) which was at the origin may move to the extremity of the radius whose length is $\sqrt{a^2 + c^2}$, and in like manner he curve $C_{m'}$ be translated so that the point (in its plane and carried with it) which was at the origin may move to the extremity of the radius whose length is $\sqrt{b^2 + c^2}$; and if then the radii be moved round the centre, maintaining their constant angle, while the curves C_m, $C_{m'}$ are made to move without rotation, the locus of the intersections of C_m, $C_{m'}$ in successive corresponding positions will be a curve of the kind in question.

To take a simple instance which can readily be verified, let ABCDEF represent a double parallel ruler, the bar AB being fixed, while CD, EF move parallel to one another and to AB. The angles CAE, DBF are equal and constant; in fact, we have two equal and similar tri-



angles ACE, BDF moving about A, B as centres, instead of the usual simple radii.

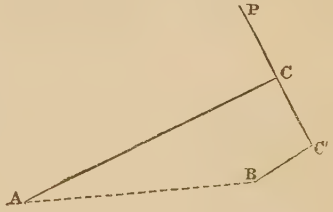
Then, if GP is a right line rigidly attached to CD, and HP is a right line rigidly attached to EF, the locus of P, their intersection, is an ellipse. We have here the locus of a point rigidly connected with a limited line which moves between two linear directrices. If we replace GP, HP by curves, we have the general locus.

It will be observed that the general forms of the equations of two curves, translated in the manner above described, are

$$\left.\begin{aligned} \phi\,(x-a\cos\theta+c\,\sin\theta,\quad y-a\sin\theta-c\cos\theta) &= 0 \\ \psi\,(x-b\cos\theta+c'\sin\theta,\quad y-b\sin\theta-c'\cos\theta) &= 0 \end{aligned}\right\} \ldots\ldots (3).$$

These equations correspond to the following construction :—

Let A be a point on C_m, B a point on $C_{m'}$. From A, B draw two parallel lines AC, BC′ = a, b respectively, and so that the line PCC′ is perpendicular to both; then x_1, y_1 being, as before, the coordinates of A ; x_2, y_2 those of B ; x, y those of P ; and writing c for PC, c' for PC′, and θ for the angle made by AC and BC′ with the axis of x ; we have



$$\begin{aligned} x_1 &= x-a\cos\theta+c\sin\theta, \\ y_1 &= y-a\sin\theta-c\cos\theta, \\ x_2 &= x-b\cos\theta+c'\sin\theta, \\ y_2 &= y-b\sin\theta-c'\cos\theta. \end{aligned}$$

Since the equations (3) represent, therefore, the locus of a point P rigidly connected with AB, which is constant, they are not intrinsically more general than the system (1).

4. If there are three equations homogeneous in x, y, z, and of the forms

$$\begin{aligned} &\mathrm{A}\,z^m+(\mathrm{B}\,x+\mathrm{C}\,y)\,z^{m-1}+(\mathrm{D}\,x^2+\mathrm{E}\,xy+\mathrm{F}\,y^2)\,z^{m-2}+\&\mathrm{c}.=0, \\ &\mathrm{A}'z^n+(\mathrm{B}'x+\mathrm{C}'y)\,z^{n-1}+(\mathrm{D}'x^2+\mathrm{E}'xy+\mathrm{F}'y^2)\,z^{n-2}+\&\mathrm{c}.=0, \\ &\mathrm{A}''z^p+(\mathrm{B}''x+\mathrm{C}''y)\,z^{p-1}+(\mathrm{D}''x^2+\mathrm{E}''xy+\mathrm{F}''y^2)\,z^{p-2}+\&\mathrm{c}.=0, \end{aligned}$$

where A, A′, A″ are of the orders μ, ν, π in uneliminated variables, B, C ; B′, C′ ; B″, C″ of the orders $\mu+\alpha$, $\mu+\beta$; $\nu+\alpha$, $\nu+\beta$; $\pi+\alpha$, $\pi+\beta$, &c., so that the orders of the coefficients proceed in arithmetical progression, the order of the resultant of the system is (Salmon's Higher Algebra, 2nd edition, p. 281)

$$mnp\left\{\frac{\mu}{m}+\frac{\nu}{n}+\frac{\pi}{p}+\alpha+\beta\right\};$$

referring to the system (2), we may put $\mu=m$, $\nu=\nu=m$, and $\alpha=\beta=-1$. Also, with respect to the third equation, we may

consider it of the type $p = \pi = 2$, $\alpha = \beta = -1$. Hence, as a major limit of the degree of the locus, we have $2mm'$; and this degree is attained in the case of two directrices made up of right lines, since the locus then consists of mm' conics.

5. In order to determine the class, we obtain the number of tangents parallel to a given line.

Writing k, a constant, for $\frac{dy}{dx}$, and substituting ϕ, ψ for the full expression of the functions, we have the equations

$$\left.\begin{aligned} &\frac{d\phi}{dx} + k\frac{d\phi}{dy} + \left\{\frac{d\phi}{dx}(a\sin\theta + c\cos\theta) + \frac{d\phi}{dy}(-a\cos\theta + c\sin\theta)\right\}\frac{d\theta}{dx} = 0 \\ &\frac{d\psi}{dx} + k\frac{d\psi}{dy} + \left\{\frac{d\psi}{dx}(b\sin\theta + c\cos\theta) + \frac{d\psi}{dy}(-b\cos\theta + c\sin\theta)\right\}\frac{d\theta}{dx} = 0 \end{aligned}\right\} (4);$$

or, eliminating $\frac{d\theta}{dx}$, and considering x, y as variables to be eliminated, we have an equation of the type

$$p = m+m'-2, \quad \pi = m+m'-1, \quad \alpha = \beta = -1;$$

the orders of the coefficients relating to $\cos\theta$, $\sin\theta$; the two equations (1) in the same sense correspond to

$$\mu = m, \quad \nu = n = m', \quad \alpha = \beta = -1.$$

Consequently the order of the resultant in $\cos\theta$, $\sin\theta$ is $mm'(m+m'-1)$; and in $\cos\theta$ or $\sin\theta$, $2mm'(m+m'-1)$. This, therefore, is the class of the locus. We get the same result by taking the order of the conditions that (1) and (4) may coexist, θ being eliminated.

Again, the order of the conditions that (1) may have two common values of θ, or that (2) may have a double set of values of X, Y, is (Salmon, loc. cit. p. 281)

$$\frac{2mm'}{2}\{2(m-1)(m'-1)+(m-1)+(m'-1)\} = mm'\{2mm'-m-m'\},$$

which gives the number of double points, or, rather, the multiplicity (including cusps, if any) of the locus.

In the present case, we have

$$2mm'(2mm'-1)-2mm'(2mm'-m-m') = 2mm'(m+m'-1).$$

Hence there are generally no cusps.

6. It would probably be difficult to apply this analytical method to the case of directrices possessing singularities; but, as in several analogous theories, there are some simple geometrical considerations which enable us to verify and generalize the preceding results.

To a double point on the directrix C_m will correspond $2m'$ double points on the locus; for a circle of given radius described about the double point as centre will cut the other directrix in $2m'$ points; and

each position of the connecting radius will give a double point on the locus. In like manner, a cusp on the directrix C_m will give rise to $2m'$ cusps on the locus. We have, of course, corresponding results relative to double points or cusps on $C_{m'}$. These conclusions are independent of the number of double points and cusps on the directrices, and hold good therefore for compound curves.

Consider, then, two directrices L_m, $L_{m'}$ made up of m and m' right lines respectively. The degree of the corresponding locus is $2mm'$, since it consists of mm' conics.

The class is also $2mm'$, but it has been reduced by the effect of $\frac{m(m-1)}{2}$ double points on L_m and $\frac{m'(m'-1)}{2}$ double points on $L_{m'}$. Therefore, for the general locus, when the directrices are without singularities, the class is

$$2mm'+4m'\frac{m(m-1)}{2}+4m\frac{m'(m'-1)}{2}=2mm'(m+m'-1).$$

If there are δ double points and κ cusps on C_m, and δ' double points and κ' cusps on $C_{m'}$, the class is

$$2mm'(m+m'-1)-4\{\delta m'+\delta' m\}-6\{\kappa m'+\kappa' m\};$$

the number of double points being

$$mm'(2mm'-m-m')+2(\delta m'+\delta' m);$$

and of cusps $\qquad 2(\kappa m'+\kappa' m).$

If n is now the class of C_m and n' that of $C_{m'}$, the class of the locus may be written $\qquad 2\{nm'+n'm+mm'\}.$

To connect the foregoing processes with the fundamental equations (1) or (2), we may observe that if C_m has a double point whose coordinates are α, β, then the first equation of (2) has a double set of roots,

$$x-aX+cY=\alpha,$$
$$y-aY+cX=\beta.$$

Combining these with the second and third equations, and considering x, y, X, Y as independent variables, we have $2m'$ corresponding double points on the locus; and similar reasoning applies to cusps. The double points and cusps are supposed not to lie on the line at infinity.*

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* Since this paper was read before the Society, I have observed, in a communication by M. Chasles to the *Comptes Rendus* for May, 1871, several results which bear upon the subject. The author gives, without actual demonstration, numerous theorems which he has obtained by the principle of correspondence. The mode of generation mentioned in §3 enables us to obtain the degree of the locus with which we are concerned by means of the correspondence of points. For consider the linear transversal L cutting the moving curves in any position. It will cut C_m in m points, and $C_{m'}$ in m' points. To a given point on L, considered as an intersection with C_m, in m of its positions, will correspond mm' points on L considered as intersections with $C_{m'}$, and *vice versâ*. Hence the number of united points, or the degree of the locus, is $2mm'$. The power of the principle is unfortunately often one-sided, and when it gives the degree with ease, yields the class with difficulty, and conversely.

Point directrices coincident.

7. If the two directrices coincide, the formula must be modified. Take a directrix consisting of m right lines. We have as the locus $2m$ right lines and $\frac{m(m-1)}{2}$ conics, which last number is to be doubled, since the ends of the moving line may be reversed. Hence we have for the general degree $2m^2$; or we must simply make $m' = m$ in the previous formula. But for the special class we have $2m(m-1)$, while each double point or cusp on the directrix gives rise to $4m$ double points or cusps on the locus, because each end of the traversing chord may be placed at the point in question. So that we have for the general class

$$2m(m-1)+8m\,.\,\frac{m(m-1)}{2}-8\delta m-12\kappa m,$$

or
$$4m^3-2m^2-2m-8\delta m-12\kappa m,$$

or
$$4mn+2m(m-1),$$

where δ and κ are the numbers of double points and cusps, and n is the class of the directrix.

The number of double points on the locus is $2m^3(m-1)+(4\delta+1)m$, and the number of cusps is $4\kappa m$. Of the double points, m are at infinity, and coincide with the points where the directrix meets the line at infinity.

Point directrices having common points at infinity.

8. If there are two distinct directrices having common points at infinity, these points are multiple points on the locus. For take a system of right lines L_m having p lines parallel to q lines of the second directrix $L_{m'}$, also composed of right lines; we shall have as part of the locus $2pq$ right lines having the same direction as the parallel lines of the directrices; that is to say, there is a multiple point of the order $2pq$ on the locus. Hence, if there are common multiple points at infinity of the orders p, q; p_1, q_1, &c., on the two directrices, the same points are multiple points on the locus of the orders $2pq$, $2p_1q_1$, &c., and the class suffers a corresponding reduction, its value being

$$2mm'(m+m'-1)-4(m\delta'+m'\delta)-6(m\kappa'+m'\kappa)-2\Sigma pq.$$

If the two directrices are of the degrees m, and have m common points at infinity, the degree of the locus is $2m^2$, and the class

$$4m^3-2m^2-2m-8\delta m-12\kappa m,$$

agreeing with the formula for a single directrix.

If the single directrix has a multiple point of the order p at infinity, the same point is of the order $2p^2$ on the locus.

Take as directrices the parabola $y^2-kx=0$ and the line $y-l=0$, we have
$$(y-a\sin\theta-c\cos\theta)^2-k(x-a\cos\theta+c\sin\theta)=0\ \ldots\ldots(a),$$
$$y-b\sin\theta-c\cos\theta-l=0\ \ldots\ldots\ldots\ldots\ldots\ldots(b).$$

From (b) we get

$$\sin\theta = \frac{b(y-l)+c\sqrt{b^2+c^2-(y-l)^2}}{b^2+c^2},$$

$$\cos\theta = \frac{c(y-l)-b\sqrt{b^2+c^2-(y-l)^2}}{b^2+c^2};$$

and, by (a) and (b),

$$\{l-(a-b)\sin\theta\}^2-k(x-a\cos\theta+c\sin\theta)=0.$$

Hence, by substitution,

$$\{(a-b)^2\{b^2(y-l)^2+c^2[b^2+c^2-(y-l)^2]\}+(kc-2lb)(a-b)(b^2+c^2)(y-l)\}^2$$
$$-(kx-l^2)(b^2+c^2)^2\}$$
$$-[b^2+c^2-(y-l)^2]\{2(a-b)bc(y-l)-[k(ab+c^2)+2bc(a-b)](b^2+c^2)\}^2$$
$$=0.$$

This is of the form

$$(Ay^2+Byz+Cxz+Dz^2)^2-(Ey^2+Fyz+Gz^2)(Hy+Kz)^2=0;$$

which has a finite double point on $Hy+Kz$, and two adjacent double points at $y=0$, $z=0$.

If the second directrix $C_{m'}$ is a translation of the first directrix C_m, and the length of the moving line is the distance of translation, the locus will contain a curve which also is identical with C_m translated; and if we exclude it, the locus is of the degree $2m^2-m$. We have a case of this when the directrices are two equal circles, and the chord is the distance between their centres. The locus then consists of a circle equal to the directrices and a nodal bicircular quartic.

Conditions of mechanical possibility.

9. The nature of the foregoing constructions suggests some general remarks. Mechanical possibility introduces obvious limitations on the analytical formulæ. These include in their form the results of mechanically impossible constructions, and these results may be real. Of course, among the trajectories of points of the moving plane, some possess singularities in excess of the numbers which we have obtained. The directrices themselves are extreme cases; but it must be remembered that the whole of a directrix cannot generally be traversed by the describing point, in this case an extremity of the finite line. Hence an arc of a directrix becomes the limit of a locus, or a degenerate curve of higher degree. The conditions may even be such that the plane is practically fixed, and each point represents a locus; or the conditions may be inconsistent, as when the length of the moving bar is less than the shortest distance between the directrices. By a special arrangement of the constants we may obtain a directrix as the locus corresponding to a possible combination, but not generally. When, however, we consider unreal combinations, the directrices are always included among the trajectories.

Thus, if we have two linear directrices, the loci arising from possible

movement are ellipses, since, except for a pair of parallel lines, the moving line cannot reach infinity. Considering, therefore, the motion of an extremity, we have a limited line on one of the directrices, representing a flattened ellipse. But, analytically, a limited line may move with a real extremity on one directrix, and an unreal one on the other; and then the real extremity moves along the whole of a directrix. In the same way, generally, while an arc of a directrix is a limiting trajectory corresponding to a possible construction, the whole of the directrix is a trajectory corresponding to unreal constructions. Practically a locus cannot have real points at infinity, unless the directrices have common points at infinity; for if one curve is at infinity finitely distant from another, the distance is analytically evanescent, and the curves meet at infinity. If the directrices have common points at infinity, but the traversing bar cannot pass continuously towards infinity, the multiple points at infinity will be conjugate.

Case of $m' = 1$.

10. There are several particular cases which merit attention. If $m=2$, $m'=1$, the locus is of the fourth degree with two double points. There are 10 disposable constants. The conic gives 5, the line gives 2; the length of the moving line, the ratio in which it is divided by the perpendicular from the describing point, and the length of that perpendicular, supply 3 more. The curves resulting from the construction are analytically subjected to only two conditions beyond those implied in the existence of the double points. There are two parallel double tangents whose points of contact lie with the two double points on a determinate conic.*

When $m'=1$, we can readily eliminate. There will be no real loss of generality in taking $y=0$ for the line, the other equation remaining general.

The system is reducible to

$$y - b\sin\theta - c\cos\theta = 0,$$
$$\phi\{x - a\cos\theta + c\sin\theta,\ (b-a)\sin\theta\} = 0.$$

From the first of these equations we get

$$\sin\theta = \frac{by + c\sqrt{b^2+c^2-y^2}}{b^2+c^2},$$

$$\cos\theta = \frac{cy - b\sqrt{b^2+c^2-y^2}}{b^2+c^2}.$$

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* For the case of a circle and line as directrices, forms of these curves are given in connection with a paper "On the Mechanical Description of some species of Circular Curves," *Proceedings*, vol. ii., p. 128.

Substituting these values in the second equation, we get

$$\phi\left\{x+\frac{(b-a)\,c+(ab+c^2)\sqrt{b^2+c^2-y^2}}{b^2+c^2},\ (b-a)\frac{by+c\sqrt{b^2+c^2-y^2}}{b^2+c^2}\right\}=0.$$

Rationalised, this equation is of the form

$$\Phi^2-\Psi^2\,(b^2+c^2-y^2)=0,$$

and shows that there are two multiple tangents parallel to the linear directrix, and having in general m points of contact, and that there are $m(m-1)$ double points which form the intersect of two curves of the mth and $(m-1)$th degrees respectively. The $2m$ points of contact of the multiple tangents, and the $m(m-1)$ double points, lie on a curve of the mth degree. The $mm'(2mm'-m-m')$ double points of the general locus are also connected in some way, and it would be interesting to know what the relation is.

When the general motion of the plane is determined, we have only two constants at our disposal, viz., a and c, or b and c. Hence the double points and cusps of all the trajectories will form loci relative to the fixed plane. I suppose that the locus of cusps will, in general, be the locus of instantaneous centres relative to the fixed plane, and will be due to points situated on the corresponding locus relative to the moving plane; but I have not determined these loci. In the case of two lines as directrices, points on the moving locus of instantaneous axes describe flattened ellipses, and not, of course, properly cusped curves; but the exception is of the kind which proves the rule, for the trajectories are doubly orthogonal on the same side to the fixed locus of instantaneous centres. The double point of a trajectory is always the intersection of the lines.

Case of one point directrix, $a=-b, c=0$.

11. There is a reduction of one-half in the order of the locus when there is only one directrix, and the describing point is the middle point of a constant chord. We can see this by taking a directrix composed of conics and lines. The degree of the locus of the middle point of a constant chord moving about a conic is 4, and the locus of the middle point of a limited line moving along an indefinite line is the latter line. Now, in the case in question, the reversing of the ends of the moving line when the directrices are two conics or lines, or a line and conic, reproduces the same locus. Hence we have not to double the order in the corresponding cases. Thus, if the directrix consist of $\frac{m}{2}$ conics, we have for the degree

$$\frac{8\frac{m}{2}\left(\frac{m}{2}-1\right)}{2}+\frac{4m}{2}=m^2;$$

or if the directrix consists of m lines, we have $m(m-1)+m=m^2$.

12. The result may be established analytically.

We have, since $a-b = 2a$,

$$\phi\,(x-a\cos\theta,\quad y-a\sin\theta) = 0,$$
$$\phi\,(x+a\cos\theta,\quad y+a\sin\theta) = 0;$$

or, putting ϕ for $\phi\,(x, y)$,

$$\left.\begin{aligned}\phi + \left\{\frac{d^2\phi}{dx^2}\cos^2\theta + \frac{d^2\phi}{dy^2}\sin^2\theta + 2\,\frac{d^2\phi}{dxdy}\sin\theta\cos\theta\right\}\frac{a^2}{1.2} + \ldots = 0\\ \left(\frac{d\phi}{dx}\cos\theta + \frac{d\phi}{dy}\sin\theta\right) + \ldots\ldots = 0 \ldots\ldots\ldots\ldots\end{aligned}\right\}\ (5).$$

The order of the system is to be determined.

Generally, if (μ) denote a coefficient whose order is μ, the system

$$(s)\ x^{2m} + (s+\alpha)\,x^{2m-1} + (s+2\alpha)\,x^{2m-2} + \ldots\ldots + [s+\alpha]\,x + [s] = 0$$
$$(t)\ x^{2n} + (t+\alpha)\,x^{2n-1} + (t+2\alpha)\,x^{2n-2} + \ldots\ldots + [t+\alpha]\,x + [t] = 0$$

may be written in the form

$$\left.\begin{aligned}(s)\,(\cos\theta, \sin\theta)_m + (s+\alpha)\,(\cos\theta, \sin\theta)_{m-1} + \ldots + (s+m\alpha) = 0\\ (t)\,(\cos\theta, \sin\theta)_n + (t+\alpha)\,(\cos\theta, \sin\theta)_{n-1} + \ldots + (t+n\alpha) = 0\end{aligned}\right\}\ \ldots(c),$$

where $(s)\,(\cos\theta, \sin\theta)_m$ means a homogeneous function of $\cos\theta$, $\sin\theta$ of the order s as to variables in the coefficients, and the order m as to $\cos\theta$, $\sin\theta$.

Combining this system with $\cos^2\theta + \sin^2\theta = 1$, which is of the type

$$(0)\,(\cos\theta, \sin\theta)_2 + (\alpha)\,(\cos\theta, \sin\theta)_1 + (2\alpha)\,(\cos\theta, \sin\theta)_0 = 0$$

when α is positive, we get for the order of the resultant ($\cos\theta$, $\sin\theta$ being eliminated)

$$2mn\left\{\frac{s+m\alpha}{m} + \frac{t+n\alpha}{n} + \alpha - 2\alpha\right\} = 2ns + 2mt + 2mn\alpha.$$

If we multiply the second of equations (c) by $Ax+B$, A and B being constants, we change the form to

$$(t)\ x^{2n+1} + (t+\alpha)\,x^{2n} + (t+2\alpha)x^{2n-1} + \ldots\ldots + [t+\alpha]\,x + [t] = 0.$$

But the order of the resultant becomes

$$2ns + 2mt + 2mn\alpha + s + m\alpha,$$

or

$$(2n+1)\,s + 2mt + (2mn+m)\alpha.$$

In like manner, for two equations of the degrees $2m+1$, $2n+1$, we get

$$(2n+1)\,s + (2m+1)\,t + (2mn+m+n)\alpha.$$

Now the system (5) may be written, when m is even, in the form

$$(0)\,(\cos\theta, \sin\theta)_m + (2)\,(\cos\theta, \sin\theta)_{m-2} + \ldots\ldots = 0$$
$$(1)\,(\cos\theta, \sin\theta)_{m-1} + (3)\,(\cos\theta, \sin\theta)_{m-3} + \ldots\ldots = 0,$$

or in the form

$$(0)\,X^m + (2)\,X^{m-1} + (4)\,X^{m-2} + \ldots\ldots + [2]\,X + [0] = 0$$
$$(1)\,X^{m-1} + (3)\,X^{m-2} + (5)\,X^{m-3} + \ldots\ldots + [3]\,X + [1] = 0,$$

and the order of the resultant is

$$m+m\,(m-1) = m^2.$$

Similarly, we get m^2 if m is odd. Consequently, the degree of the locus is m^2.

12. The class can be determined from the consideration that a double point or cusp on the directrix gives $2m$ double points or cusps on the locus. If we take a system of m right lines as the directrix, the locus consists of the lines themselves and $\frac{m\,(m-1)}{2}$ conics. The class is therefore $m\,(m-1)$. But, allowing for the effect of the double points, we obtain for the general class

$$m\,(m-1)+2m^2\,(m-1) = 2m^3-m^2-m\,;$$

and if there are δ double points and κ cusps, the class is

$$2m^3-m^2-m-(4\delta+6\kappa)\,m, \quad \text{or} \quad 2mn+m\,(m-1),$$

where n is the class of the directrix.

Case of one directrix, $a=b$.

13. Let us suppose that the directrices are identical, and that the describing point is on the perpendicular to the moving line at its middle point.

The system (1) becomes

$$\phi\,(x-a\cos\theta+c\sin\theta,\ \ y-a\sin\theta-c\cos\theta) = 0,$$
$$\phi\,\{x-(a-k)\cos\theta+c\sin\theta,\ \ y-(a-k)\sin\theta-c\cos\theta\} = 0,$$

k being evanescent; and if, further, we now suppose a to become evanescent, we have, by expansion, dividing by k, and then making a and $k=0$,

$$\phi = \phi\,(x+c\sin\theta,\ \ y-c\cos\theta) = 0,$$
$$\frac{d\phi}{d\theta} = 0.$$

In fact, the moving line becomes infinitesimally small, the locus becomes an envelope, and is a parallel of the directrix.

Referring to the remarks of § 3, it is plain that, in the case under discussion, we have the moving radii equal and coincident, the describing curves coincide in one, and the locus is, in fact, the envelope of a curve moving parallel to itself about a circle. This envelope is, as I have elsewhere remarked, a parallel of the moving curve.

But the parallels are described only by points of the moving plane lying at any moment on a normal to the directrix; that is to say, on the perpendicular to the infinitesimal moving element. It is desirable to pay attention to the general locus, in connection with the special case in which the locus is a parallel, and has an exceptional interest of its own.

14. Imagine a straight line moving tangentially about a directrix $\phi(x, y) = 0$. We have to treat of the locus of the extremity of a perpendicular to the tangent, having its foot always at a constant distance from the point of contact, the length of the perpendicular being of course given.

The equations (1) become in this case

$$\phi = \phi(x - a\cos\theta + c\sin\theta,\ y - a\sin\theta - c\cos\theta) = 0 \ \ldots\ldots\ (a'),$$

$$\frac{d\phi}{dx}\cos\theta + \frac{d\phi}{dy}\sin\theta = 0 \ \ldots\ldots\ (b').$$

The directrix being free from singularities, the degree of the locus remains $2m^2$, and is the same as for parallels.

To find the class, we have, putting $\frac{dy}{dx} = k$,

$$\left\{\frac{d\phi}{dx}(c\cos\theta + a\sin\theta) + \frac{d\phi}{dy}(c\sin\theta - a\cos\theta)\right\}\left\{\frac{d\theta}{dx} + \frac{d\phi}{dx} + k\frac{d\phi}{dy}\right\} = 0,$$

or, by the second equation of condition,

$$a\left\{\frac{d\phi}{dx}\sin\theta - \frac{d\phi}{dy}\cos\theta\right\}\left\{\frac{d\theta}{dx} + \frac{d\phi}{dx} + k\frac{d\phi}{dy}\right\} = 0 \ \ldots\ldots\ (c');$$

and differentiating the second equation, we get

$$\left\{\begin{array}{l}\left[\frac{d^2\phi}{dx^2}(c\cos\theta + a\sin\theta) + \frac{d^2\phi}{dx\,dy}(c\sin\theta - a\cos\theta)\right]\cos\theta \\ \quad + \left[\frac{d^2\phi}{dx\,dy}(c\cos\theta + a\sin\theta) + \frac{d^2\phi}{dy^2}(c\sin\theta - a\cos\theta)\right]\sin\theta \\ \qquad - a\left[\frac{d\phi}{dx}\sin\theta - \frac{d\phi}{dy}\cos\theta\right]\end{array}\right\}\frac{d\theta}{dx}$$

$$+ \frac{d^2\phi}{dx^2}\cos\theta + \frac{d^2\phi}{dx\,dy}\sin\theta + k\left\{\frac{d^2\phi}{dx\,dy}\cos\theta + \frac{d^2\phi}{dy^2}\sin\theta\right\} = 0 \ \ldots\ (d').$$

If now we eliminate from (b'), (c'), (d') the quantities $\frac{d\theta}{dx}$, $\cos\theta$, $\sin\theta$, so far as they are explicitly contained, we shall obtain an equation of the degree $2m(m-1)$ in $(x - a\cos\theta + c\sin\theta)$, $(y - a\sin\theta - c\cos\theta)$, say X, Y; and combining this with (a'), we have for the number of tangents parallel to a given line $2m(m-1)$. The two equations give, in fact, the points of the directrix corresponding to such parallel tangents.

Substituting X, Y in the expressions, we may write the equations

$$\left.\begin{array}{l}\phi(X, Y) = 0, \\ \left\{(a + ck)\frac{d\phi}{dY} + (c - ak)\frac{d\phi}{dX}\right\}^2 \\ \quad \times\left\{\left(\frac{d\phi}{dY}\right)^2\frac{d^2\phi}{dX^2} - 2\frac{d\phi}{dX}\frac{d\phi}{dY}\frac{d^2\phi}{dX\,dY} + \left(\frac{d\phi}{dX}\right)^2\frac{d^2\phi}{dY^2}\right\}^2 \\ \quad - a^2\left\{\left(\frac{d\phi}{dX}\right)^2 + \left(\frac{d\phi}{dY}\right)^2\right\}^3\left(\frac{d\phi}{dX} + k\frac{d\phi}{dY}\right)^2 = 0\end{array}\right\}\ldots(e').$$

If R be the radius of curvature at an intersection, the equation gives

$$\frac{(a+ck)\frac{d\phi}{dY}+(c-ak)\frac{d\phi}{dX}}{a\left(k\frac{d\phi}{dY}+\frac{d\phi}{dX}\right)}=\pm R,$$

an expression which can be geometrically interpreted.

If $a=0$, the factor dependent on k is $\left(\frac{d\phi}{dX}+k\frac{d\phi}{dY}\right)^2=0$, giving $2m(m-1)$ for the class in the case of parallels. It is plain, in this case, that the points of the directrix with which we are concerned lie upon the first polar of the point at infinity which determines the direction of the tangents, and that each such point on the directrix corresponds to two tangents of the parallel.

15. It will be observed, that since

$$2m^2(2m^2-1)-8m(m-1)=4m^4-10m^2+8m,$$

and cusps do not enter the general locus, we may take the number of double points to be $2m^4-5m^2+4m$.

Now, in the case of parallels,

$$\text{number of double points}=2m^4-11m^2+10m,$$
$$\text{number of cusps}=6m^2-6m;$$
$$\text{so that the multiplicity}=2m^4-5m^2+4m,$$

as in the general case above treated.

We may take it, then, that a certain number of double points, viz. $6m^2-6m$, are converted into cusps when the locus becomes an envelope relative to its two equations.

The foregoing result is confirmed by the consideration that the oblique parallels (as they may be called) and the parallels have a (1, 1) correspondence.*

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* The result is accordant with a theorem mentioned by Dr. Salmon ("Cambridge and Dublin Mathematical Journal," Vol. III., p. 170, note), viz., the curve obtained by eliminating t between

$$at^m+mbt^{m-1}+\frac{m(m-1)}{2}ct^{m-2}+\ldots=0,$$
$$a't^m+mb't^{m-1}+\frac{m(m-1)}{2}c't^{m-2}+\ldots=0,$$

where the coefficients are of the first order in the variables x, y, z, is of the $2m$th degree having the maximum number of double points; but if $a'=b$, $b'=c$, &c., then, the total number of double points remaining the same, the maximum number of these $[=3(m-1)]$ will become cusps. Under the limitations specified by Dr. Salmon, we have an envelope relative to t as parameter. There seems to be some more general theorem on the subject.

The reduction in the degree of the locus on account of δ double points and κ cusps on the directrix, is $4\delta+6\kappa$, as in the case of parallels. In both instances, a circle described about a double point or cusp fulfils the geometrical conditions, and by parity of reasoning the circle has to be taken twice for a double point, and three times for a cusp, in both cases. In fact, a double point evidently gives rise to a doubled circle. If we take a very small loop, and draw the corresponding locus by points, it immediately appears that another circle becomes part of the locus at the limit. The general degree therefore is $2(m+n)$, both for oblique and proper parallels.

The class for a conic is 16, and for two lines it is zero. The reduction for the double point is therefore 16; and since the effect is local, as in the case of proper parallels, the class remains $8n$.

The case of cusps requires special consideration. I do not find that a cusp on the directrix gives rise to cusps on the oblique parallel, and therefore conclude that, since the multiplicity (in this instance the deficiency also) is the same as in proper parallels, the class is $8n+2\kappa$.

In fact, reverting to the equation which determines by its intersections with the directrix the number of parallel tangents, let us take as the directrix the semicubical parabola $x^3-py^2=0$.

We have $$X^3-pY^2=0,$$

$$\{3(c-ak)X^2-2(a+ck)pY\}^2\{4p^2Y^2.6X-2p.9X^4\}^2$$
$$-a^2\{9X^4+4p^2Y^2\}^3\{3X^2-2kpY\}^2=0.$$

If we put in this $X^{\frac{3}{2}}$ for $\sqrt{p}Y$, it becomes divisible by X^{11}, or, when rationalized, by X^{22}. Hence the reduction on account of the cusp at the origin is 22, in conformity with the value $8n+2\kappa$.

More generally, we may take the curve $x^s-py^{s-1}=0$. The resulting equation contains the factor $X^{\left(8s-4-6\frac{s}{s-1}\right)}$, and, when rationalized, the factor $X^{8s^2-18s+4}$.

But the origin in this case being equivalent to $s-2$ cusps and $\frac{(s-2)(s-3)}{2}$ double points, we have

$$22(s-2)+16\frac{(s-2)(s-3)}{2}=8s^2-18s+4.$$

16. The principal conclusions up to this point are summed up in the following synoptic form. I denote the degree, the class, the number of double points, and the number of cusps of the directrix C^n_m, by m, n, δ, κ, and the corresponding characteristics of the directrix $C^{n'}_{m'}$ by the same letters accented.

Two Directrices.		One Directrix.			
			$b=-a,\ c=0$	$b=a$	$b=a=0$ (Parallel)
Order	$2mm'$	$2m^2$	m^2	$2(m+n)$	$2(m+n)$
Class	$2(mn'+m'n+mm')$	$4mn+2m(m-1)$	$2mn+m(m-1)$	$8n+2\kappa$	$2n$
Double points	$mm'(2mm'-m-m')+2(m\delta'+m'\delta)$	$2m^3(m-1)+(4\delta+1)m$	$\frac{m^4-2m^3+m}{2}+2\delta m$	$2(m+n)^2-5(m+n)+4m-\kappa$	$2(m+n)^2-11(m+n)+10m-3\kappa$
Cusps	$2(m\kappa'+m'\kappa)$	$4\kappa m$	$2\kappa m$		$6n+2\kappa$

It is supposed that the line at infinity has no special relation to the directrices or directrix, as the case may be.

Envelopes in the moving plane.

17. I now pass to the question of envelopes. The envelope of a line which rests with a given interval on two directrices is of the class $4mm'$, where m, m' denote the degrees of the directrices. In fact, Professor Cayley has pointed out, that if two points on two given curves C_m, $C_{m'}$ have an (α, α') correspondence, the class of the envelope of the line joining those points is $m\alpha' + m'\alpha$. In the present case $\alpha = 2m$ and $\alpha' = 2m'$, since a circle of given radius about a point on the first directrix as centre meets the second directrix in $2m'$ points. The class is independent of double points or cusps on the directrices. The degree, however, is affected by these singularities. To each double point or cusp on the first directrix correspond $2m'$ double tangents, or stationary tangents on the envelope. Hence, if there are δ double points and κ cusps on C_m, and δ' double points and κ' cusps on $C_{m'}$, we shall have $2(m\delta' + m'\delta)$ double tangents, and $2(m\kappa' + m'\kappa)$ stationary tangents on the envelope. The degree of the envelope may now be determined, as follows:

Take two directrices consisting of m and m' right lines respectively. For a pair of lines the degree is 6 and the class 4. The degree of the envelope in the present case is therefore $6mm'$; and correcting this for the effect of the $\frac{m(m-1)}{2}$ double points on C_m, and the $\frac{m'(m'-1)}{2}$ double points on $C_{m'}$, we have

$$6mm' + 2mm'(m'-1) + 2m'm(m-1) = 2mm'(m+m'+1);$$

and if there are δ double points and κ cusps on C_m^n, and δ' double points and κ' cusps on $C_{m'}^n$, we have to deduct $4(m\delta' + m'\delta) + 6(m\kappa' + m'\kappa)$, and we may write the result $2(mn' + m'n + 3mm')$. In the case of a pair of lines, there are 6 cusps and 4 double points, 3 double tangents, and no stationary tangents. Double points on the directrices do not introduce stationary tangents. Hence, for general directrices, there are

$2(m\kappa' + m'\kappa)$ stationary tangents (or inflexions),

$8m^2m'^2 - mm'(m+m'+3) + 2(m\delta' + m'\delta)$ double tangents,

$2(mn' + m'n + 3mm')^2 - (mn' + m'n + 3mm') - 2mm' - 9mm'(m+m'-1) - 3(m\kappa' + m'\kappa)$ double points, and

$6mm'(m+m'-1) + 2(m\kappa' + m'\kappa)$ cusps.

I am indebted to Mr. Cotterill for the following determination of the degree of the envelope when a right line and curve are directrices:—

Corresponding to each intersection of the line with the curve there will be two contacts of the envelope with the line. Also for each intersection of the line by the parallel of the curve (mod. length of chord) we have an intersection of the envelope and line. Hence the whole

number of intersections, *i.e.* the degree of the envelope, is $4m+2(m+n)$, which agrees with the general formula.

The degree may be shown analytically as follows for two directrices without singularities:

The point of contact of the chord with its envelope is the foot of the perpendicular let fall on it from the intersections of the normals to the directrices at its extremities. If we write X, Y for the coordinates of the intersection of the normals, x, y for those of the point of contact, and in other respects follow our former notation, we get the following conditions:

$$\phi(x_1, y_1) = 0, \quad \psi(x_2, y_2) = 0,$$

$$(X-x_1)\frac{d\phi}{dy_1} - (Y-y_1)\frac{d\phi}{dx_1} = 0,$$

$$(X-x_2)\frac{d\psi}{dy_2} - (Y-y_2)\frac{d\psi}{dx_2} = 0,$$

$$(x_1-x_2)^2 + (y_1-y_2)^2 = k^2,$$

$$(y-Y)(y_1-y_2) + (x-X)(x_1-x_2) = 0,$$

$$(y-y_1)(x_1-x_2) - (x-x_1)(y_1-y_2) = 0.$$

Since the axes are arbitrary, it is sufficient to find the degree of the resultant in x or y, when x_1, y_1, x_2, y_2, X, Y are eliminated.

By eliminating X, Y and then x, y successively, we get a system corresponding to

Order in y	Order in x_1, y_1, x_2, y_2
1	$m+m'+1$
0	$m+m'+1$
0	m
0	m'
0	2

and the order of the resultant in y is therefore $2mm'(m+m+1)$.

The envelope will in general have $2m'(m+n)$ contacts with C_m^n, and $2m(m'+n')$ contacts with $C_{m'}^{n'}$.

These conclusions relate to the moving limited line itself. It remains to consider the envelope of a line rigidly connected with the primary moving line.

18. It is known that, if a limited line move about two linear directrices, the envelope of a parallel to the moving line is a parallel curve to the envelope of the moving line itself, and breaks up into two curves possessing characteristics like those of the envelope.

We may consider it, therefore, as a curve of the 12th degree with a multiplicity 56, one more than $\frac{11 \cdot 10}{2}$, the limit for a proper curve. Making use of the same reasoning as before, we may pass to the case of a limited line moving on two directrices $C_m, C_{m'}$. We have for the general degree and class of the envelope of a line parallel to the moving line $4mm'(m+m'+1)$ and $8mm'$ with $12mm'(m+m'-1)$ cusps, and

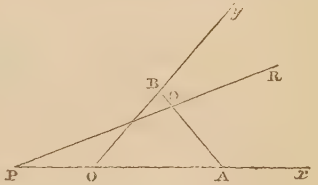
$8m^2m'^2(m+m'+1)^2-20mm'(m+m'+1)+32mm'$ double points. The multiplicity is therefore not always in excess of the limit for a proper curve, but we remark that it is precisely sufficient for two curves of the degree $2mm'(m+m'+1)$ and class $4mm'$ with $6mm'(m+m'-1)$ cusps and $2m^2m'^2(m+m'+1)^2-10mm'(m+m'+1)+16mm'$ double points. It is natural to conjecture that the parallel curve is compound in the general case. In fact, taking account only of the curve which corresponds to $+d$, d being the distance of the generating line from the primary line, the whole reasoning appears applicable to the curve thus generated; and we have for a pair of lines the degree 6 and class 4; for two distinct pairs of lines as directrices, the degree 24 and class 16; and therefore, for two conics, the degree 40 and class 16, and so on.

I apprehend that when we are considering the envelope of a line in the moving plane parallel to the line joining the two points whose motions determine the motion of the plane, we must consider d, the distance between the lines, as persistently $+$ or $-$, and therefore that the degree and class of the corresponding envelope are $2mm'(m+m'+1)$ and $4mm'$.

The corrections for double points and cusps on the directrices will be similar to those for the preceding case. In fact, the general formulæ remain the same.

19. We shall now be able to deal with the case in which the enveloping line divides the moving chord in a given ratio, and makes with it a given angle.

Take, in the first instance, two linear directrices OAx, OBy, inclined at an angle $AOB=\omega$. Let AB be a position of the moving bar; PQR a line dividing AB in Q so that $AQ=a$, $BQ=b$; and let $\angle BAx=\alpha$, $\angle PQA=\phi$.



The coordinates of Q are

$$\frac{b\sin(\alpha-\omega)}{\sin\omega},\quad \frac{a\sin\alpha}{\sin\omega};$$

and by the formula for the tangent of the angle between two lines referred to oblique coordinates, the equation of PQR is of the form

$$\left(y-\frac{a\sin\alpha}{\sin\omega}\right)+\mathrm{K}\left(x-\frac{b\sin(\alpha-\omega)}{\sin\omega}\right)=0,$$

where

$$\mathrm{K}=\frac{\sin\alpha\sin(\phi-\omega)-\sin\phi\sin(\alpha-\omega)}{\sin\alpha\sin\phi-\sin(\alpha-\omega)\sin(\phi+\omega)}.$$

The result is of the form

$$\mathrm{A}\cos\alpha+\mathrm{B}\sin\alpha+\mathrm{C}\cos 2\alpha+\mathrm{D}\sin 2\alpha+\mathrm{E}=0,$$

where A, B are of the first order, and C, D, E are of the order zero relative to x and y.

The envelope of the line is consequently of the 6th order, and has six cusps and four double points. This corresponds to the parallel when one sign of d is taken.

A double point or cusp on the directrix C_m appears to produce $2m'$ double tangents, or stationary tangents on the envelope; and, *mutatis mutandis*, the same is true for $C_{m'}$. Hence, if the two directrices are made up of m lines and m' lines respectively, the degree is $6mm'$; and adding on account of the double points $2mm'(m+m'-2)$, we have for the general degree $2mm'(m+m'+1)$. The class is $4mm'$. We get, therefore, the same general formulæ as in the particular cases just treated of.

If there is one directrix only, the class and degree of the envelope of a line require to be modified. Taking a system of m lines, the class of the envelope is $4m^2-2m$, and is unaffected by the double points. This therefore is the general class. There will be, in general, $4\kappa m$ stationary tangents, and the deficiency appears to be the same as for a trajectory of a point connected with a constant chord. I infer, therefore, that the degree is $4m^3+2m^2-6m-8\delta m-12\kappa m$, a formula which, however, does not hold for the chord itself.*

20. The envelope of a curve traced on the moving plane is, in general, the same as the envelope of the trajectories of the points of the curve. ("Nouvelles Annales," April, 1869.)

For if an envelope with respect to a, b, c as parameters is derived from

$$\chi(a, b, c) = 0,$$
$$\phi(x, y, a, b, c) = 0,$$
$$\psi(x, y, a, b, c) = 0,$$

it is immaterial as to nett result whether we eliminate a, b, and take the discriminant as to c; or eliminate a, c, and take the discriminant as to b.

But if a, c, in the system (1), are coordinates relative to the moving plane, the equation $\chi(a, c) = 0$ represents a curve χ traced on it. If we combine this with (1), eliminate θ, and take the discriminant as to a or c after eliminating the other letter, we get the envelope of the trajectories of points on $\chi(a, c) = 0$. If we eliminate a, c, we get the equation of χ referred to the fixed plane for a given value of θ; and taking the discriminant as to θ, we get the envelope of the curves χ. The cases of simple sliding or simple rolling are exceptional. [See "Nouvelles Annales," t. vii. (Laisant), where the problem is treated geometrically.]

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* For the chord, the class reduces to $2m^2-2m$, exclusive of m points at infinity.

If there is a common point of the order p on C_m and q on $C_{m'}$, the class is $4mm'-2pq$, and the degree is $2\{mn'+m'n+3(mm'-pq)\}$.

In the case of the chord, there are still further reductions when there are common asymptotes.

21. If an envelope is derived from

$$\chi(a, b, c) = 0,$$
$$\xi(x, y, z;\ a, b, c) = 0,$$

the functions being homogeneous in x, y, z, and in a, b, c, the degree of the envelope in x, y, z is reduced, in consequence of a double point or cusp on $\chi(a, b, c) = 0$ regarded as a point curve, by $2m_1$, or $3m_1$, where m_1 is the degree of the second equation in x, y, z. For if a_1, b_1, c_1 are simply double values of a, b, c satisfying the first equation, the curve $\xi(x, y, z;\ a_1, b_1, c_1) = 0$ is consecutive to itself, and counts twice in the envelope which it enters as a factor. Hence the degree of the nett envelope is reduced by $2m''$. The result for a cusp follows similarly.

We are now supposing that the functions are general, except as expressly conditioned, at least in so far that a, b, c enter into the second equation in a manner entirely independent of the form of χ.

We have found that the envelope derived from $Aa+Bc+C=0$ and the system (1) is of the degree $2mm'(m+m'+1)$. Let us take a series of m'' lines on the moving plane, the degree of the envelope is then $2mm'm''(m+m'+1)$; and adding on account of the double points $2 \cdot 2mm' \dfrac{m''(m''-1)}{2}$, we get, for the general degree,

$$2mm'm''(m+m'+m'').$$

If the traced curve is a circle, its envelope will evidently be a parallel of the trajectory described by the centre. Hence its degree is $2\{2mm' + 2mm'(m + m' - 1)\} = 4mm'(m+m')$, and its class $4mm'(m+m'-1)$.

I obtain for the general class

$$2mm'(m+m'-1)\,m''(m''-1)+4mm'm''.$$

If the degree is reduced by $2m_1$ or $3m_1$ for a double set a_1, b_1, c_1, it equally follows that the class is reduced by $2n_1$ or $3n_1$ under the same conditions. The general theorem is of wide application, and worth, I think, further investigation.*

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* As a mere verification, take Prof. Henrici's result that the degree and class of the envelope of a unicursal series of curves $(\lambda)^r (x_1x_2x_3)_m = 0$ are respectively $2m(r-1)$ and $m\{3mr-2(m+r)+2\}$. If the parametric locus consists of p linear factors, there are $p-1$ double points in excess of the number for a unicursal locus.

The degree then for a unicursal locus is

$$2mp(r-1)+2m(p-1) = 2m(pr-1).$$

The class will be

$$mp\{3mr-2(m+r)+2\} + 2m(m-1)(p-1) = m\{3mpr-2(m+pr)+2\};$$

these are the proper values, since the index of the system of curves is pr.

For a general parametric locus without double points or cusps, we have, in like manner,

$$\text{Degree} = 2mp(r-1)+mp(p-1) = mp\{2r+p-3\},$$
$$\text{Class} = m(m-1)\{p^2+3p(r-1)\}+mpr.$$

For δ double points and κ cusps on the parametric locus, the corrections are $-m(2\delta+3\kappa)$ and $-m(m-1)(2\delta+3\kappa)$. There are $3m^2p(p+r-3)-6m^2\delta-8m^2\kappa$ cusps. The above expression for the degree agrees with a known result relative to the Tact Invariant of two curves. (See Salmon's *Higher Algebra*, 2nd Ed., p. 154.)

With respect to the motion of a plane in plane space, however conditioned, the envelopes of parallel lines must form a system of involutes having a common evolute. We are therefore at liberty to select the line of the set which will give the simplest result.

In general, an envelope of a line, and a trajectory of a point, have a (1, 1) correspondence. If two curves have a (1, 1) correspondence, their deficiency is the same. This correspondence may relate to points, or lines, or points of one and lines of the other. (Clebsch, Crelle, t. lxiv. p. 98, quoted by Cayley, Quarterly Journal, vol. xi. p. 185.)

We have seen that the double and stationary tangents of the envelope of a line are in number

$$8m^2m'^2-mm'(m+m'+3)+2\{m(\delta'+\kappa')+m'(\delta+\kappa)\},$$

and the class is $4mm'$. Therefore

$$\text{Deficiency of Envelope} = mm'(m+m'-3)-2\{m(\delta'+\kappa')+m'(\delta+\kappa)\}+1$$

$$=\frac{(2mm'-1)(2mm'-2)}{2}-\{mm'(2mm'-m-m')+2[m(\delta'+\kappa')+m'(\delta+\kappa)]\}$$

= Deficiency of Trajectory.

Circular point directrices.

22. When the directrices pass through the circular points at infinity, there are cases of peculiar difficulty. I shall endeavour to determine the formulæ when those points are points of simple multiplicity.

If the order of multiplicity is p as to C_m and q as to $C_{m'}$, the degree of the trajectory seems to be $2mm'-2pq$. By the condition $\cos^2\theta+\sin^2\theta=1$, the equations (1) are, in the case supposed, reducible to the orders $m-p$, $m-q$, respectively, relative to $\cos\theta$ and $\sin\theta$, while they remain of the orders m, m' in x and y. They are of the types

$$(p)\,(\cos\theta,\sin\theta)_{m-p}+(p+1)\,(\cos\theta,\sin\theta)_{m-p-1}+\ldots=0,$$

$$(q)\,(\cos\theta,\sin\theta)_{m'-q}+(q+1)\,(\cos\theta,\sin\theta)_{m'-q-1}+\ldots=0;$$

where $(p)\,(\cos\theta,\sin\theta)_{m-p}$ means, as formerly, a homogenous function of $\cos\theta$, $\sin\theta$ of the order $m-p$, whose coefficients are of the order p in x, y. Adding to these the equation $\cos^2\theta+\sin^2\theta-1=0$ of the type $(0)\,(\cos\theta,\sin\theta)_2+(1)\,(\cos\theta,\sin\theta)_1+(2)=0$,

we have, as before, for the order of the resultant relative to θ,

$$2(m-p)(m'-q)\left(\frac{m}{m-p}+\frac{m'}{m'-q}+1-1-1\right)=2mm'-2pq.$$

This result may be verified by means of compound directrices; C_m consisting, say, of p circles and $m-2p$ lines, and $C_{m'}$ of q circles and $m'-2q$ lines. The degree of the locus is

$$2(m-2p)(m-2q)+4p(m'-2q)+4q(m-2p)+6pq=2mm'-2pq;$$

for in the case of two circles as directrices the degree is 6, and in the

case of a line and circle the degree is 4. The double points do not affect the degree.

23. The more difficult question of the class remains. When only one of the directrices (say C_m) passes through the circular points, the degree remains unaltered. But we cannot infer the same relative to the class. A double point or cusp on C_m gives rise to $2m'$ double points or cusps on the locus. I apprehend, however, that a double point or cusp on $C_{m'}$ only gives rise to $2(m-1)$ double points or cusps on the locus, the circular points being extraneous since they correspond to every position of the extremity of the traversing chord which lies upon $C_{m'}$.*

If this be correct, we have for the class of the trajectory, by means of compound directrices consisting of lines,

$$2mm'+2(m-1)m'(m'-1)+2m'm(m-1),$$

or

$$2mm'(m+m'-1)-2m'(m'-1).$$

When $m'=1$, the class is the same, whether C_m is circular or not; and this agrees with the result obtained for a circle and line as directrices, in which case the trajectory is a circular quartic with two double points.

Again, if we assume this last conclusion, and consider C_m as made up of p circles and a general curve of the degree $m-2p$, while $C_{m'}$ consists of m' lines, we have for the class

$$2m'(m-2p)^2+8m'p,$$

and for the general case we must add the correction

$$2(m-p)m'(m'-1)+4m'\{2p(m-2p)+p(p-1)\},$$

giving for the class

$$2mm'(m+m'-1)-4m'p(p-1)-2pm'(m'-1).$$

If there are δ finite double points, and κ finite cusps on C_m, δ' finite double points and κ' finite cusps on $C_{m'}$, there is a reduction of

$$4\{(m-p)\delta'+m'\delta\}+6\{(m-p)\kappa'+m'\kappa\}.$$

The circular points are multiple in the degree pm'.

Taking now two sets of m lines and m' lines L_m, $L_{m'}$, the one containing p pairs of circular asymptotes, the other containing q such pairs. The trajectory consists of $mm'-2pq$ conics, and $2pq$ parallel lines. The class is therefore $2mm'-4pq$. But to obtain the general class, we

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* In determining the number of normals which can be drawn to a curve from a given point, we meet with a similar case. For a point at infinity, there are in general $m+n$ normals, n finite, and m lying at infinity; but if the curve is circular, there are only $m-2$ normals at infinity. As M. Chasles has remarked (*Comptes Rendus*, April, 1871), the reduction is due, not simply to the tangents at the circular points being also normals, but to the fact that *every* line through a circular point is normal to the curve extraneously.

must correct for $\frac{m(m-1)}{2} - p(p-1)$ double points on L_m, and for $\frac{m'(m'-1)}{2} - q(q-1)$ on $L_{m'}$ (these being the numbers of the double points exclusive of the circular points). Hence we have

$$2mm' - 4pq + 4(m-p)\left\{\frac{m'(m'-1)}{2} - q(q-1)\right\} + 4(m'-q)\left\{\frac{m(m-1)}{2} - p(p-1)\right\}.$$

Or we may avoid the foregoing imaginary combinations as follows:— For two circles as directrices I find that the trajectory is of the degree 6, and has the circular points for triple points, and three other finite double points, giving the class 12.

Hence, if C_m consists of p circles and $m-2p$ lines, and $C_{m'}$ of q circles and $m'-2q$ lines, the class is

$$12pq + 8q(m-2p) + 8p(m'-2q) + 2(m-2p)(m'-2q).$$

For the general class we must add the correction for double points

$$4(m-p)\left\{\frac{(m'-2q)(m'-2q-1)}{2} + 2q(m'-2q) + q(q-1)\right\} + 4(m'-q)\left\{\frac{(m-2p)(m-2p-1)}{2} + 2p(m-2p) + p(p-1)\right\},$$

and the result is the same as before, viz.

$$2mm'(m+m'-1) - 2pm'(m'-1) - 2qm(m-1) - 4mq(q-1) - 4m'p(p-1) + 4pq(p+q-3).$$

If the double points, exclusive of circular points, on C_m, $C_{m'}$ are δ, δ', and the cusps κ, κ', there is a reduction of

$$4\{(m-p)\delta' + (m'-q)\delta\} + 6\{(m-p)\kappa' + (m-q)\kappa\}.$$

The circular points at infinity are of the order $pm' + qm - pq$.

In the several cases, if there are no cusps on the directrices, there are none on the locus. Hence the number of cusps, when they exist, is known by their origin from cusps on the directrices, and Plücker's equations give the several characteristics.

24. Let us now consider the case in which the two circular directrices coincide.

Take p circles and $m-2p$ lines, the degree of the locus is

$$2p + 2 \cdot 4p(m-2p) + 6p(p-1) + 2(m-2p)(m-2p-1) + 2(m-2p).$$

Since a circle gives the degree 4, the locus being two concentric circles; a circle and line give 4; two circles give 6; and two lines give 2. Except for circles and lines taken singly, the numbers have to be doubled. Hence the degree is $2m^2 - 2p(p+1)$; and this is the general degree, because the double points do not affect it.

As to the class, we have, for a compound directrix consisting of p circles and a general curve C_{m-2p},

$$4p+12p\,(p-1)+\{4\,(m-2p)^3-2\,(m-2p)^2-2\,(m-2p)\}$$
$$+\ 2p\,\{2\,.\,2\,(m-2p)(m-2p+1)-2\,(m-2p)(m-2p-1)\},$$

and for the general correction

$$4\,.\,2\,(m-p)\{2p\,(m-2p)+p\,(p-1)\}\,;$$

taking into account double points, not circular points.

The result is

$$4m^3-2m^2-2m-4pm\,(m-1)-8mp\,(p-1)+8p^3-12p^2-4p.$$

This formula, however, presents difficulties which I have not been able to remove, and it may require some modification. There seems to be a difference in respect to double points situate at the intersection of tangents at the circular points. The class for a circle as directrix being 4, if we now suppose it to break up into a pair of circular asymptotes, the locus consists of two imaginary circles, and the class remains 4. In fact, a circle of given radius about the double point as centre meets the circular asymptotes at points adjacent to the circular points at infinity; and all these intersections seem to be extraneous in some way.

Accordingly, if we take m lines, containing a pair of circular asymptotes p times over, we have for the class

$$4\left\{\frac{m\,(m-1)}{2}-p\,(p-1)\right\}+8\,(m-p)\left\{\frac{m\,(m-1)}{2}-p\,(p-1)\right\},$$

giving $4m^3-2m^2-2m-4pm\,(m-1)-8mp\,(p-1)+8p^3-12p^2+4p.$

But, in consequence of the p real intersections of circular asymptotes, this result is in excess by $8p$, the last term being $+$ instead of $-$.

25. When $a=b$, and the sole directrix passes p times through the circular points, we have, by taking the special case of p circles and a curve of the degree $(m-2p)$,

$$\text{Order} = 2\,(m-2p)^2+4p.$$

But the $2p\,(m-2p)+p\,(p-1)$ double points not at the circular points have reduced the order by $4\,\{2p\,(m-2p)+p\,(p-1)\}$; the general expression therefore is $2m^2-4p^2$, or $2\,(m+n)-4p$ if n is the class, an expression which holds good when there are double points and cusps besides the multiple circular points.

To obtain the class, we have in the first place, for the compound directrix, $8\,(m-2p)(m-2p-1)+4p,$

to which add the correction for the effect of double points not circular points, $16\,\{2p\,(m-2p)+p\,(p-1)\}\,;$

and we have $8\,\{m\,(m-1)-2p\,(p-1)\}-12p = 8n-12p.$

In the corresponding case for parallel curves, we have $6n-12p$ cusps,

and if these are changed into double points, the class is changed to $8n-12p$. If there are κ cusps, the class will be $8n-12p+2\kappa$.

Whenever the line at infinity is specially related to the directrix, reductions similar to those in the corresponding cases for parallels may be looked for with respect to oblique parallels. Thus, if the directrix touches the line at infinity p times, there is a reduction of the degree by $2p$. The degree being the same as that of the corresponding parallel, and the multiplicity being the same, we have, because there are in general no cusps, $8n+2k-6p$ for the class.

Thus, for the parabola $y^2-px=0$, the system of equations is

$$(y-a\sin\theta-c\cos\theta)^2-p\,(x-a\cos\theta+c\sin\theta)=0,$$
$$2\,(y-a\sin\theta-c\cos\theta)\sin\theta-p\cos\theta=0.$$

But the first of these may be written by means of the second,

$$(y-a\sin\theta-c\cos\theta)\cos\theta-2\sin\theta\,(x-a\cos\theta+c\sin\theta)=0;$$

and the elimination is reducible to that of t from equations of the form

$$A_0t^4+B_1t^3+C_0t^2+D_1t+E_0=0,$$
$$A_0't^4+B_1't^3+C_0't^2+D_1't+E_0'=0,$$

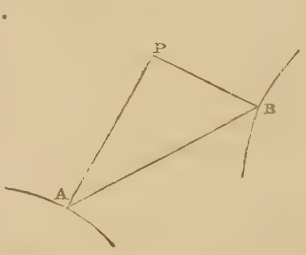
where the suffixes denote the orders of the coefficients in x, y. The resultant is therefore of the degree 6. Otherwise, writing for the first equation $p\cos^2\theta-4\sin^2\theta\,(x-a\cos\theta+c\sin\theta)=0$,

we see that y appears in the degree 6, and x in the degree 4. But since $x=0$ meets the line at infinity at a point infinitely distant from the parabola, it appears that 6 is the degree of the locus. There are two adjacent double points where the directrix touches the line at infinity.

Again, substituting $p^{\frac{1}{2}}X^{\frac{1}{2}}$ for Y in the equation (e') we obtain an equation of the 8th degree in X. But there are two tangents altogether at infinity. Hence the class is 10, and the oblique parallel is a unicursal sextic.

Locus of instantaneous centres relative to the fixed plane. Point directrices.

26. The instantaneous centre for a given position of the moving line AB is the intersection of the normals PA, PB to the directrices C_m, $C_{m'}$ at the extremities A, B of the moving chord.



Hence if $\phi(x, y)=0$, $\psi(x, y)=0$ are the equations of the directrices, we have the conditions

$$(x_1-x_2)^2+(y_1-y_2)^2=k^2,$$
$$\phi(x_1, y_1)=0,\quad \psi(x_2, y_2)=0,$$

$$(x-x_1)\frac{d\phi\,(x_1, y_1)}{dy_1} - (y-y_1)\frac{d\phi\,(x_1, y_1)}{dx_1} = 0,$$

$$(x-x_2)\frac{d\psi\,(x_2, y_2)}{dy_2} - (y-y_2)\frac{d\psi\,(x_2, y_2)}{dx_2} = 0.$$

In order to obtain the equation to the locus of the instantaneous centres, we must eliminate x_1, y_1, x_2, y_2.

Since the axes are general and do not necessarily meet the locus at infinity, it will be sufficient to determine the highest power of x or y in the resultant.

But eliminating successively x and y from the last two equations, we have an equation y in the first power without x, and another equation containing x in the first power without y. The other variables enter in the degree $m+m'-1$ in both equations. The order of the resultant in x or y is $2mm'(m+m'-1)$, and this, therefore, is the degree of the locus of instantaneous centres relative to the fixed plane. This result may be verified by means of compound directrices. If there is a double point on C_m, every line through it may be regarded as a double normal in the same sense that every such line may also be regarded as a double tangent. It follows that $2m'$ normals to the directrix $C_{m'}$ are included in the locus, and are to be counted twice over. The double point, therefore, causes a reduction of $4m'$ in the degree of the locus. So also a double point on $C_{m'}$ causes a reduction of $4m$. When the double point becomes a cusp, the reductions in the corresponding cases are, by similar reasoning, $6m'$ and $6m$.

For a system of m lines, and a system of m' lines, the degree is $2mm'$, since two linear directrices give a circle. But this degree has suffered reduction on account of $\frac{m(m-1)}{2}$ double points on C_m and $\frac{m'(m'-1)}{2}$ double points on $C_{m'}$. We must for the general degree add, therefore, $4m'\frac{m(m-1)}{2}+4m\frac{m'(m'-1)}{2}$, and the result is $2mm'(m+m'-1)$. The reduction for δ double points and κ cusps on C_m, and δ' double points and κ' cusps on $C_{m'}$ will be $4(m\delta'+m'\delta)+6(m\kappa'+m'\kappa)$. I infer also that the circular points will be multiple and of the order mm'. When the directrices are a right line and curve, it is evident that the intersections of the locus with the line will be those of the line and a parallel of the curve. Hence the degree of the locus is $2m+2n$, which agrees with the general expression $2(mn'+m'n+mm')$.

27. In order to obtain the class, I make use of the following considerations.

We have seen that the reduction of order on account of a double point or cusp on C_m is $4m'$ or $6m'$, m' being the degree of the other directrix. That is to say, the reduction is independent of the degree of

C_m and also of its other double points or cusps; in fact, independent of its general form and characteristics, except at the singularity in question.

I apprehend we may, in the case before us, extend this conclusion to the reduction of class on account of a double point on a directrix, and consider that each corresponding double position of the moving chord gives rise to a reduction $-pm$ or $-pm'$ as the case may be, so that taking the class relative to two systems of m right lines and m' right lines, and adding the correction $p\left\{\frac{m(m-1)}{2}.2m'+\frac{m'(m'-1)}{2}.2m\right\}$, we have, for the general class

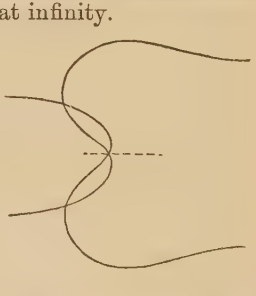
$$2mm'+p\,.\,mm'\,(m+m'-2),$$

where the numerical multipler p has to be determined.

To determine p, we ought to take the case in which $C_{m=2}$ is a conic, and $C_{m'=1}$ a line; and if the class is q, a double point on C_2 reduces the class to 4, and we have an equation to determine p. But this case, simple as it is, gives an equation of the degree 8 in a form which renders it very difficult to determine the singularities with the necessary precision. I am led therefore to make use of the more general consideration, that the effect of a double point or cusp on C_m in reducing the class, is independent of the form elsewhere of C_m, and remains the same when it passes through the circular points at infinity.

Now if we take as directrices a circle and line, the equation of the circle being $x^2+y^2-r^2=0$, and that of the line $x-a=0$, the equation of the locus of instantaneous centres relative to the fixed plane is without difficulty found to be

$$(x^2+y^2)(y^2-k^2+r^2+a^2)^2-4r^2\,(ax+y^2)^2=0.$$



This curve is of the 6th degree, having a double point at the origin, two double points where the lines $y^2-k^2+r^2+a^2=0$ meet the line $ax+k^2-r^2-a^2=0$; and, further, a quadruple point at infinity on $y=0$. The figure represents a form of the curve when two branches meeting the line at infinity on $y=0$ are imaginary.

The equivalent of double points is 9, and the class 12. If we make $r=0$, the class becomes zero. I infer that, generally, if the second directrix is of the order 1, the reduction is 6.2; and for $C_{m'}$, $6.2m'$. It follows that for two general directrices the class is

$$2mm'+6mm'(m+m'-2)=2mm'(3m+3m'-5).$$

I suppose, from analogy, that a cusp will produce $\frac{3}{2}$ of the effect of a double point, and that for δ double points and κ cusps on $C_{m'}$, δ' double points and κ' cusps on $C_{m'}$, the correction is

$$-12\,(\delta m'+\delta' m)-18\,(\kappa m'+\kappa' m).$$

It would seem that since, when there are no double points or cusps

on the directrices, one point P of a trajectory corresponds to one point P′ of the locus of instantaneous centres relative to the fixed plane, and *vice versâ*, the curves have a (1, 1) correspondence, and consequently their deficiencies should be the same. The deficiency for a general trajectory is $mm'(m+m'-3)+1$, and this is the same as

$$\frac{\{2mm'(m+m'-1)-1\}\{2mm'(m+m'-1)-2\}}{2}$$

$$-\tfrac{1}{2}\{2mm'(m+m'-1)[2mm'(m+m'-1)-1]-2mm'(3m+3m'-5)\},$$

the deficiency for the locus in question.

28. We have next to determine the degree of the locus of instantaneous centres relative to the moving plane.

The quantities a, c, in (1), may now be taken as current coordinates; so that, following the same constructions and notation as before, we have the following equations of condition:

$$\begin{aligned}
x-x_1 &\equiv a(x_2-x_1)-c(y_2-y_1),\\
y-y_1 &\equiv a(y_2-y_1)+c(x_2-x_1),\\
x-x_2 &\equiv (a-k)(x_2-x_1)-c(y_2-y_1),\\
y-y_2 &\equiv (a-k)(y_2-y_1)+c(x_2-x_1);
\end{aligned}$$

and substituting these values in the equations of the normals at x_1, y_1, x_2, y_2, which pass through x, y, we have for our system of equations

$$\{a(x_2-x_1)-c(y_2-y_1)\}\frac{d\phi(x_1,y_1)}{dy_1}-\{a(y_2-y_1)+c(x_2-x_1)\}\frac{d\phi(x_1,y_1)}{dx_1}=0,$$

$$\{(a-k)(x_2-x_1)-c(y_2-y_1)\}\frac{d\psi(x_2,y_2)}{dy_2}$$

$$-\{(a-k)(y_2-y_1)+c(x_2-x_1)\}\frac{d\psi(x_2,y_2)}{dx_2}=0,$$

$$\phi(x_1,y_1)=0,\quad \psi(x_2,y_2)=0,$$

$$(x_2-x_1)^2+(y_2-y_1)^2=k^2.$$

We cannot, as in the previous case, reduce the order by the artifice of eliminating a and c successively. Whether we do this or not, the degree of the resultant is $2mm'(m+m')$ in a, c.

A difficulty was raised in my mind as to this conclusion, because, in the case of two right angles as directrices, the locus is generally taken to be a circle; AB then subtending a constant angle at P. But it will be remarked that the locus in this case is really a circle taken twice over, since to each position and form of the triangle APB corresponds another position and equal and similar form of the triangle.

As in the case of the locus relative to the fixed plane, a double point on C_m causes the locus to contain $2m'$ lines doubled. Taking a system of m lines and a system of m' lines as directrices, the degree of the

corresponding locus is $4mm'$. We obtain for the general degree, by adding the correction for double points

$$4\left(m.\frac{m'(m'-1)}{2}+m'.\frac{m(m-1)}{2}\right),$$

the expression, as before, $2mm'(m+m')$;

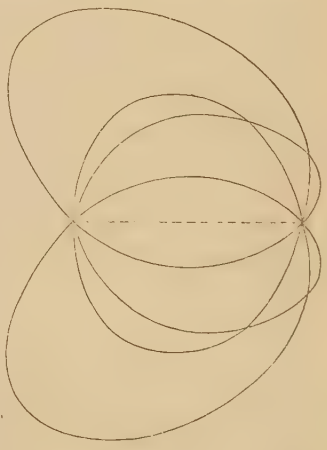
and using the former notation for double points and cusps on the directrices, we have a further correction

$$-4(m\delta'+m'\delta)-6(m\kappa'+m'\kappa).$$

29. To obtain the class, I follow the same course as in obtaining that of the locus relative to the fixed plane. Taking a circle and line for the directrices, the equations being $x^2+y^2-r^2=0$, $x-a=0$, I find for the locus now in question

$$\{[(x-k)^2+y^2][k^2x^2+a^2(x^2+y^2)]-r^2(x^2+y^2-kx)^2\}^2$$
$$-4a^2k^2(x^2+y^2)x^2[(x-k)^2+y^2]^2=0.$$

The curve has a quadruple point at the origin. The point $(x-k=0,$ $y=0)$ is another quadruple point. Besides these, we have two pairs of adjacent or united double points at the circular points at infinity, and two double points where $x=0$ meets the line $(a^2-r^2)y^2+a^2k^2=0$. We have now accounted for the intersections of the curves represented by the squared factors equated to zero. But there are two more double points determined by $x^2+y^2=a^2=kx$. (See figure.) In all, then, there exists the equivalent of 20 double points, and the class is 16. If $r=0$, the class becomes zero. This reduction therefore is 16, and I infer that if $C_{m'}$ is general, the corresponding reduction on account of a double point on C_m is $16m'$.



Hence, assuming, as in the previous case, that the class is of the form

$$4mm'+p\left\{m'.\frac{m(m-1)}{2}+m.\frac{m'(m'-1)}{2}\right\},$$

we have $p=16$, and the class is

$$2mm'(4m+4m'-6).$$

If we suppose, as I think is probably the case, that a cusp produces $\frac{3}{2}$ of the effect of a double point, we get the general correction, on account of singularities on the directrices,

$$-16(\delta m'+\delta' m)-24(\kappa m'+\kappa' m).$$

30. It is, unfortunately, difficult to verify these formulæ relating to instantaneous centres, on account of the high numbers which arise from the simplest cases. If we take as directrices the conic $Ax^2+By^2+C=0$ and the line $x-a=0$, I see no reason why the special relation of the line to the conic should reduce the class. But in this case I find the equation of the locus of instantaneous centres relative to the moving plane in the form

$$\{[AB(x^2+y^2-kx)^2+k^2A^2y^2][k^2x^2+a^2(x^2+y^2)] + CB(x^2+y^2-kx)^2(x^2+y^2)\}^2 - 4a^2k^2(x^2+y^2)x^2[AB(x^2+y^2-kx)^2+k^2A^2y^2]^2=0.$$

This is of the 12th degree. It appears to have an octuple point at the origin, and a quadruple point at $(x-k=0,\ y=0)$. We have also two double points where $x=0$ meets the lines $ABa^2y^2+A^2k^2a^2+CBy^2=0$, and at the circular points at infinity two quadruple points and two pairs of adjacent double points. There are, moreover, two double points at $x^2+y^2=a^2=kx$. In all, therefore, we have the equivalent of 54 double points, and the class is $12\,.\,11-2\,.\,54=24$, which is also the class given by the formula on putting $m=2$, $m'=1$.*

When $a=b=0$, the locus of instantaneous centres relative to the fixed plane is the evolute of the directrix, and the locus relative to the moving plane is the line constantly normal to the directrix. An involute, in fact, is generated by a point on an indefinite line rolling on the evolute. The circumstance that cusps of a parallel correspond to double points of an oblique parallel, is in harmony with this; for in the one case the generating point is on the rolling line, and gives rise to cusps when that point comes upon the evolute, while in the other case the generating point is off the rolling line.

31. If we eliminate θ from the system (1) as before, but consider x, y as determined quantities, while a, c are variables, we may regard the latter quantities as coordinates, relative to the moving plane, of a fixed point on the fixed plane. In other words, considering now the former moving plane as fixed, and the former fixed plane as moving, subject to the conditions of the construction, we have, in the present way of viewing the result of the elimination of θ, the equation of the corresponding trajectory.

The degree of the equation is $4mm'$. The effect of a double point or

* There is obviously much which remains to be investigated. Circular directrices can be treated as in the case of a trajectory of a point. When there is only one directrix, the degrees of the loci seem to be $2m^3-m^2-m$ and $2m^3-2m$; but the class is difficult to determine.

I take the opportunity to correct an oversight at the end of § 10. The double point in question may be anywhere on the plane. It appears to follow that the double points of trajectories are distributed over the whole plane, on account of the manner in which the parameters a, c enter their equations.

cusp on C_m (now a moving directrix) is to give rise to $2m'$ double points or cusps on the locus, and similarly with respect to singularities on $C_{m'}$. Hence, taking compound directrices in the moving plane, consisting of m lines and m' lines, we have the class $4mm'$. Correct now for the effect of the double points, and we have in the general case $2mm'(m+m')$. If there are δ double points and κ cusps on C_m, and δ' double points aud κ' cusps on $C_{m'}$, the further correction is $-4(m\delta'+m'\delta)-6(m\kappa'+m'\kappa)$. The number of cusps is $2mm'+2m\kappa'+2m'\kappa$, each of the circular points counting as mm'.

The deficiency is the same as for the original trajectory.

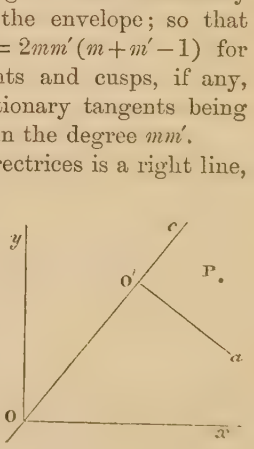
It is plain that, if lines AC, BC, containing a constant angle, move about the points A, B respectively, any point P, rigidly connected with them, will describe a Limaçon of Pascal of the degree and class 4, and having the circular points at infinity for cusps, and a finite double point. Any line rigidly connected with AC, BC envelopes a circle. In point of fact, it easily appears that if a constant angle circumscribes two circles, the locus of the apex, or of any point in the plane of the angle moving with it, is a Limaçon.

If there is only one directrix C_m, the corresponding degree of the trajectory of a point of its plane is $4m^2-2m$, and the class is

$$4m^3-4m-8m\delta-12m\kappa.$$

As to the envelope of a line in the moving plane, the class for directrices consisting of m lines and m' lines is $2mm'$; and since the double points and cusps on the directrices generally give rise to no double points or cusps on the envelope, I take the general class to be $2mm'$. But double points and cusps on the directrices give rise immediately to double tangents and stationary tangents on the envelope; so that we have, in general, $2mm'+2mm'(m+m'-2)=2mm'(m+m'-1)$ for the degree, with a reduction for double points and cusps, if any, $-4(m\delta'+m'\delta)-6(m\kappa'+m'\kappa)$, the number of stationary tangents being $2(m\kappa'+m'\kappa)$. The circular points are multiple in the degree mm'.

Taking the particular case when one of the directrices is a right line, we may proceed as follows :—



Let Ox, Oy be rectilinear axes in the fixed plane. Let the linear moving directrix be $OO'c$; and let $O'a$, $O'c$ be rectilinear axes at O' in the moving plane. Putting $\angle O'Ox=\theta$, $OO'=\rho$, we have for any point P of the moving plane

$$a=x\sin\theta-y\cos\theta,\quad c=x\cos\theta+y\sin\theta-\rho.$$

Hence, for a line $c-ma=0$, we have

$$(\cos\theta-m\sin\theta)\,x+(\sin\theta+m\cos\theta)\,y-\rho=0.$$

Put L for $\dfrac{\cos\theta-m\sin\theta}{\rho}$, M for $\dfrac{\sin\theta+m\cos\theta}{\rho}$, so that these may

be taken as tangential coordinates of the line relative to the fixed plane. Then $\sin\theta = \rho\,\frac{M-mL}{1+m^2}$, $\cos\theta = \rho\,\frac{mM+L}{1+m^2}$, $\rho^2 = \frac{1+m^2}{L^2+M^2}$.

But for a curve $\phi\,(a, c) = 0$ in the moving plane constantly passing through $(y=0, x=b)$, we have $\phi(b\sin\theta,\ b\cos\theta-\rho) = 0$, or

$\phi\,\{b\,(M-mL),\ b\,(mM-L)-(1+m^2),\ \surd(1+m^2)\surd(L^2+M^2)\} = 0$.

The class, therefore, of the envelope of $c-ma=0$ is $2m$, when m is the degree of the curve. There are $m\,(m-1)$ double tangents, giving $2m^2$ for the degree.

32. Whatever may be the motion of a plane in plane space, it is clear that, if two trajectories of its points are given, any third may be described by means of a limited line moving between the given trajectories and rigidly connected with the describing point. But, in general, not only is the length of the limited line determined, but we cannot take all its positions. We can only conclude that the third trajectory in question is a part of the general locus. The length of the limited line must, of course, be the distance between the points which describe the given trajectories.

The kinematical properties of a plane involve much more than the determination of the degrees and characteristics of trajectories and envelopes. The correspondence extends to arcs, areas, and curvature. The particular study of these relations is, however, practically distinct from the subject of the present paper, though both lines of investigation would no doubt meet in a complete theory.

To facilitate the further study of the subject, it would be desirable to work out in detail the elementary cases. I am not aware that the simple case of two linear directrices has been elaborated; but probably a pretty complete account could be made up out of existing materials.

In conclusion, I may say that, while some of the methods which I have employed require great caution in their application, I believe them to be fundamentally sound. It is, of course, essential to solve accurately the different problems with reference to the elementary combinations. Any error in these particulars may lead to extravagantly wrong results; and I am not without apprehension that some of my conclusions may be faulty from this cause. But mistakes of this sort will not invalidate the principles used, which may, indeed, be probably carried much further. We may take systems of points instead of systems of lines, and *vice versâ*, or even employ combined systems of points and lines. I trust, however, that whatever may be thought of the means employed, some of my results possess interest.

The cases next in natural order seem to be the motion of a plane determined by the envelopes of two of its lines or by the locus of a point and the envelope of a line. I have obtained some general formulæ

relating to these motions which may perhaps be worth more detailed discussion than I am able to give them at present. The motions of points and lines may be determined in an infinite number of ways, so that when the directrices are algebraical, the general motion shall be so too. Thus we may suppose two lines to be normal to two directrices, &c.* A more general case in which this holds is the motion determined by a curve sliding between given curves. The trajectories and envelopes will be algebraical when the given curves are so. This kind of motion is widely different in this respect from rolling, which under the simplest algebraical conditions generates transcendental curves.

Discussions took place on both communications.

Prof. Henrici exhibited card-board models of two ellipsoids, a hyperboloid of one sheet, and of an elliptic paraboloid; also stereograms of the models of surfaces exhibited at former meetings of the Society.

A brief discussion arose on the subject of Prof. Cayley's letter to the President (see Proceedings at the May meeting, p. 279); but it was determined that no definite action should be taken in the matter at the present time.

The following presents were then made to the Society, or subsequently received:—

Copies of the "Cambridge Mathematical Journal" for May, 1838; February and November, 1839; May and November, 1843; February, May, and November, 1844; February and May, 1845; "Quarterly Journal of Pure and Applied Mathematics," No. IV., March, 1855: from the President.

"Sulle trasformazioni razionali nello spazio," nota 1ª del Prof. L. Cremona, read May 4th, 1871: from the Author.

"Monatsbericht," April, June, and July, 1871.

"Proceedings of the Royal Society," Vol. XIX., Nos. 128, 129.

"Annali di Matematica," Serie IIª, Tom. IV., Fasc. 2, January, 1871.

"Journal of London Institution," Nos. 6, 7.

"Journal of Institute of Actuaries," July, 1871, No. LXXXIV.

"Annuaire de l'Académie Royale des Sciences, des Lettres, et des

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* I have already alluded to the communication from M. Chasles to the *Comptes Rendus* for May 1871, from which I quote the following:—

"67. Le lieu d'un point d'où on peut mener à deux courbes n'', n''', deux tangentes faisant un angle de grandeur donnée dans un sens de rotation déterminé est une courbe de l'ordre $2nn'$ qui a deux points multiples de l'ordre nn' aux deux points circulaires à l'infini."

I had found that, for a point rigidly connected with the revolving angle, the trajectory is of the degree $4nn'$, and class $2(nn' + mn' + 2nn')$, where m, m' are the degrees of the directrices. The circular points are cuspidal in the order nn'. But for the apex the order and class reduce to $2nn'$ and $nn' + n'm + 2nn'$, and the circular points are multiple in the order nn'.

Beaux Arts de Belgique," 1871; and "Bulletins de l'Académie Royale ..... de Belgique," 39me Année, 2me Série, Tomes 29, 30, 1870: from the Academy.

"Annual Report of the Board of Regents of the Smithsonian Institution for 1869": from the Smithsonian Institution.

"Crelle's Journal," 73 Band, drittes und viertes Heft.

"Appendix to Benjamin Anderson's Journey to Musada," New York, 1870; and "Transactions of the Connecticut Academy of Arts and Sciences," Vol. I., Part 2, 1867–71, New Haven: from the Academy.

"Verzeichniss der Abhandlungen der Königlich-Preussischen Akademie der Wissenschaften von 1710–1870, in Alphabetischer Folge der Verfasser," Berlin, 1871.

"Rapport sur les progres de Géométrie," par M. Chasles: from the President.

Model of the Surface called the "Cylindroid": from Prof. R. Stawell Ball.

[This is the model of a conoidal cubic surface which is presented in the Theory of the Geometrical Freedom of a Rigid Body. An abstract of a paper read before Section A of the British Association at Edinburgh, August, 1871, has been communicated by the Author to the "Philosophical Magazine and Journal of Science" for September, 1871. From this we learn that the name above given was suggested by Prof. Cayley, and that the equation to the surface is

$$z(x^2+y^2)-2axy=0.$$

In the model the parameter a is 2·6 inches. The wires which correspond in the model with the generating lines of the surface represent the axes of the screws. The distribution of pitch upon the generating lines is shown by colouring a length of $2{\cdot}6\times\cos 2\theta$ inches upon each wire. The distinction between positive and negative pitches is indicated by colouring the former red and the latter black.]

APPENDIX.

In a paper, read before the British Association Meeting, in August, 1871, Mr. Merrifield followed out the investigation of which he gave the first brief outline on pp. 222, 223. It is published also in "The Messenger of Mathematics" for October, 1871, pp. 81—88.

In connection with Mr. Todhunter's Historical Note (p. 232), the Secretaries have been favoured with the following references, pointed

out by a member of the Society, Art. Trigonometrie, Klügel's Mathem. Wörterbuch, vol. v., p. 211 :—

"Diese höchst merkwürdigen Formeln hat Gauss (Theoria Motus, etc., 1809) ohne Beweis bekannt gemacht und sie als solche, die in den Lehrbüchern noch fehlen, besonders empfohlen. Man nennt sie daher in Deutschland gewöhnlich die Gaussischen Gleichungen. Zu bemerken ist aber, dass Mollweide schon im November Stück der Monatlichen Correspondenz für das Jahr 1808, pp. 398, 399, dieselben Formeln bewiesen, und dabei zugleich bemerkt hat, dass sie in den Systemen der Trigonometrie noch fehlten, obwohl sie eine bequeme Art eine Seite, ohne einen der an dieselbe anliegenden Winkel zu kennen, zu finden, an die Hand gehen. Auch bemerkt Mollweide a. a. O. ausdrücklich dass in der dortigen Bezeichnung $\varphi = \frac{1}{2}(B-C)$, $\psi = \frac{1}{2}(B+C)$ sei, wodurch die dortigen Formeln augenblicklich in die ihnen oben gegebene Gestalt übergehen.

"Ausserdem hat noch Delambre in der *Connaissance des Tems* von 1808, dieselben Formeln bekannt gemacht, weshalb Französische Schriftsteller sie gewöhnlich nach ihm benennen."

Mollweide's Formulae, as given in Zach's Monatliche Correspondenz, are

$$\sin\frac{a}{2} = \frac{\sin\frac{1}{2}(b+c)\sin\frac{A}{2}}{\cos\varphi} = \frac{\sin\frac{1}{2}(b-c)\cos\frac{A}{2}}{\sin\varphi}, \quad \varphi = \frac{B-C}{2};$$

$$\cos\frac{a}{2} = \frac{\cos\frac{1}{2}(b+c)\sin\frac{A}{2}}{\cos\psi} = \frac{\cos\frac{1}{2}(b-c)\cos\frac{A}{2}}{\sin\psi}, \quad \psi = \frac{B+C}{2}$$

$$\sin\frac{A}{2} = \frac{\cos\frac{1}{2}(B+C)\cos\frac{a}{2}}{\cos\xi} = \frac{\cos\frac{1}{2}(B-C)\sin\frac{a}{2}}{\sin\xi}, \quad \xi = \frac{c+b}{2};$$

$$\cos\frac{A}{2} = \frac{\sin\frac{1}{2}(B+C)\cos\frac{a}{2}}{\cos\nu} = \frac{\sin\frac{1}{2}(B-C)\sin\frac{a}{2}}{\sin\nu}, \quad \xi = \frac{b-c}{2}.$$

In the course of the discussion on Prof. Crofton's Communication on Girders (p. 233), it appeared that Culmann's work ("l'excellente *Graphische Statik* de M. Culmann," Prof. Cremona in Reprint from "Educational Times," vol. xv. p. 33,) was but little known in this country.

At the instance of the Hon. Secretary, Prof. Henrici has kindly drawn up the following sketch of its contents, and of the methods of solution employed in it :—

Professor Culmann, in his "Graphische Statik," has undertaken to treat the most general statical problems geometrically, using modern Synthetic Geometry, instead of Analysis, as the instrument of investigation. Modern Geometry stands in the same relation to Graphical Statics as Analysis does to Statics as usually treated.

It is true that geometrical solutions for special problems have been long extant. But these have been either of a very elementary character —making use especially of the Parallelogram of Forces—or they appeared as results of Analysis, afterwards translated into the language of Geometry. The *result* was constructed, but too often no geometrical proof existed, so that all connection between the data and the solution was lost. To this latter class especially belong many of the results of the First Napoleon's chief engineer, though Poncelet himself has done so much for the development of Geometry that with him begins a new epoch in Geometry.

To engineers Culmann's work will be of the greatest service. The language of the engineer is drawing. His drawing materials he has always at hand, and he is ready at any moment to make use of them.

Culmann's work begins with a chapter headed "Graphical Calculation." Here the most fundamental operations with lines are considered—Addition, Multiplication, Division, Evolution, and Involution. For the latter the logarithmic spiral is used, which takes in every respect the place of a table of logarithms. Practical methods are given for the reduction of quantities of more dimensions, as areas, volumes, and especially moments, to lines. An area, for instance, is represented by the altitude of an equal rectangle, with a unit base.

Next comes the application of these principles to Statics. Forces acting upon rigid bodies in a plane or in space are combined, whereby couples appear as infinitely small forces acting in lines at infinity. From this representation, at once natural and fertile, the well-known properties of couples, as well as their combinations with forces and with other couples, follow immediately.

One of the most interesting points of this first part (Graphical Statics proper) is the investigation of moments of higher order, especially of moments of inertia. The properties of the ellipsoid, and more especially of the ellipse of inertia, are developed in a very elegant manner, and practical methods for their construction are given. In connection with this subject, Culmann introduces a new figure, which he calls the "Kern," the core, of a solid or plane figure. Its meaning is this. Suppose we have a beam bent under the influence of external forces. Then a section of the beam has a neutral fibre, and the elastic forces acting upon this section are proportional to their distance from this neutral fibre, whilst the line of application of the resultant of the elastic forces cuts the section in some finite or infinite point.

The positions of this point P and the neutral fibre are connected in the following manner:—Take a point Q symmetrical to P with regard to the centre of gravity of the section; that is to say, such that the line PQ is bisected at the centre of gravity. Then the neutral fibre of the section corresponding to the resulting force acting at P is the polar of

Q with regard to the central ellipse of inertia. Now suppose the neutral fibre moves in the plane of the section, but under the restriction never to cut a given figure, then the point P too will move, and will always remain within another figure. It is this latter figure which is called the "kern," in case the neutral fibre is excluded from the section of the beam itself. The importance of this "kern" is seen from an example. Instead of the beam, take a vertical column. Then, in order that no part of the column may be subjected to tension, it is necessary that the neutral fibre of any section should not cut this section. Hence the resultant force must always cut the "kern" of the section. If the column be built of bricks, and this condition be not satisfied, one side would be subjected to tension, and therefore be liable to be cracked. The same holds for arches.

The remaining, and greater, part of the work treats of the applications of Graphical Statics to practical problems. First, the "beam" is treated in the most general manner. The elastic forces acting upon any element of it are constructed. Among others, the curves are determined in a beam of constant section, along which no shearing forces act. The elements of the beam along such a line are therefore subjected to forces only which act either in the direction of the tangent to the curve, and produce tension or compression in the layer, or normal to the layer. In any beam there exist two systems of such curves, which intersect at right angles. The author remarks that one might cut up the beam along the curves of the one system, those which are subjected to lateral compression, without weakening the beam.

The applications to open frameworks are simple, and give results which coincide with Prof. Clerk-Maxwell's reciprocal diagrams. In the case of continuous girders, graphical methods are not sufficient, and here calculation steps in, in order to determine the *external* forces acting upon the beam. The theory of arches and suspension bridges becomes very simple and beautiful. With regard to the latter, we may mention a new investigation. To make suspension bridges rigid, light girders are generally added to the roadway, but these have always been too light, losing therefore very soon all power of resistance. Culmann now asks, how strong must these be, in order that they may effectually resist the fluctuations of the chains. He finds that the top and bottom beams of these girders have to be about ·6 times as strong as in the case where no chains are used.

The work treats, in the last chapters, very fully of the theory of the Pressure of Earth in connection with retaining walls and tunnels. The theory of Earth Pressure is that to which the author at first applied geometrical methods, and the elegant results here obtained induced him to treat Graphical Statics as a special Science.

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